Brouwer Fixed Point Theorems and Their Applications

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Abstract

The main objective of this research paper is to study the Brouwer's fixed point theorems and their relative theorems. First, we introduced the Brouwer fixed point theorem in $\mathbb{R}$. Secondly, we study the homeomorphisms and the fixed point property in metric space. Next, we focused on Brouwer fixed point theorem, which states that a fixed point exists provided X is a compact and convex subset of $\mathbb{R}_{+}^{N+1}$. After that, the contraction mapping theorem imposes a strong continuity condition on f but only very weak conditions on X. Finally, proof of the Tarski Fixed Point Theorem is mentioned.

Introduction

This paper is an exposition of the Brouwer Fixed-Point Theorem of topology and the Three Points Theorem of transformational plane geometry. If we consider a set X and a function $f: X \to X$, a fixed point of f is a point $x \in X$ such that $f(x) = x$. Brouwer's Fixed-Point Theorem states that every continuous function from the n-ball B^n to itself has at least one fixed point. An isometry is a bijective function from $\mathbb{R}^2$ to itself which preserves distance. Although the Three Points Theorem is not itself a fixed-point theorem, it is a direct consequence of the following fixed-point theorem. An isometry with three non- collinear fixed points in the identity. The Three Points Theorem states that if two isometries agree at three non-collinear points, they are equal.

Brouwer Fixed Point Theorem and Its Relative Theorems

Most of the definitions and theorems provided in this paper are referred from [1].

1. The Brouwer Fixed Point Theorem in $\mathbb{R}$

1.1 Definition

Let X be a compact convex set. A fixed point of a continuous function f :X→X is a point $\xi \in$ X satisfying $f(\xi) = \xi$.

1.2 Theorem. If X = $[a,b] \subseteq$ P and f **:**X→X is continuous then f has a fixed point.

Proof : If $f(a) = a$ or $f(b) = b$ then we have done. Otherwise, $f(a) > a$ and $f(b) < b$.

Define $g(x) = f(x) - x$. Then $g(a) > 0$ while $g(b) < 0$. Moreover, g is continuous since f iscontinuous. Therefore, by the *intermediate value theorem*, there is an $\xi \in (a,b)$ such that $g(\xi) = 0$, hence $f(\xi) = \xi$.

The following are examples in which one of the *sufficient conditions* in theorem 1.2are violated and no fixed point exists.

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1.3 Example. Let X = [0, 1) and $f(x) = \frac{x+1}{2}$. Here, f is continuous and X is connected, but X is *not compact*. Thus, f has no fixed point in [0, 1).

1.4 Example. Let X = [0, 1] and $f(x) = 1$ if $x < \frac{1}{2}$, $f(x) = 0$ if $x \geq \frac{1}{2}$.

There is no fixed point. Here X is connected and compact but f is *not continuous*.

1.5 Example. Let X = $[0,\frac{1}{3}]\cup[\frac{2}{3}, 1]$ and $f(x) = \frac{1}{2}$. There is no fixed point. Here f is continuous and X is compact, but X is *not connected*.

Finally, we notice that the conditions theorem 1.2 are sufficient but not necessary. The requirement that a fixed point must exist for *every continuous f imposes much stronger conditions* on X than the requirement that a fixed point exists for some given *f*. For given X and *f*, a fixed point may exist as long as X is not empty, even if every other condition is violated.

1.6 Example. Let X = $(0,\frac{1}{3}) \cup (\frac{2}{3},\infty)$. Let $f(x) = \frac{1}{4}$ if $x = \frac{1}{4}$, $f(x) = 0$ otherwise. Then X is not closed, not bounded, not connected and f is not continuous. But f has a fixed point, namely $\xi = \frac{1}{4}$.

Now we consider that how to generalize theorem 1.2 from a statement about $X \subseteq \mathbf{P}$ to a statement about $X \subseteq P^N$. There are two issues.

The first issue is that the line of proof in theorem1.2 does not generalize to higher dimensions. For X = [0,1], a fixed point occurs where the graph of f crosses the45°line. Since the 45°line bisects the square $[0, 1]^2 \subseteq P^2$, if f is continuous then its graph must cross this line, the proof based on the Intermediate Value Theorem formalizes exactly this intuition. In contrast, if $X = [0, 1]^2 \subseteq P^2$, then the graph of$f$ lies in the 4-dimensional cube $[0, 1]^2 \times [0, 1]^2$, and the analog of the 45° line is a 2-dimensional plane in this cube.

A 2-dimensional plane cannot bisect a 4-dimensional cube (just as a 1-dimensional line cannot bisect a 3-dimensional cube). Brouwer must, among other things, insure that the graph of*f*, which is 2-dimensional, does not spiral around the 2-dimensional 45° plane, without ever intersecting it.

The second issue is that it is not obvious how to generalize the condition that X be an interval. Requiring X to be a closed rectangle is too strong. On the other hand, requiring X to be compact and connected is too weak, as the next example illustrates.

1.7 Example. Let X be a disk with a central hole cut out $X = \{x \in P^2 : \|x\| \in [\varepsilon , 1]\}$where $\varepsilon \in (0,1)$. Then X is compact and connected but it is *not convex*.

Let f be the function that, in effect, rotates X by a half turn. More formally, represent P^2 in polar coordinates: a point (r, θ) corresponds to $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$.

Then $f(r,\theta) = (r, \theta + \pi)$. This function is continuous but it has no fixed point. Here, X is connected but not convex, which leads naturally to the conjecture that a fixed point exists if X is *compact and convex*. This intuition is correct, but convexity can be weakened, at essentially no cost, for a reason discussed in the next section.

2 Homeomorphisms and the Fixed Point Property

2.1 Definition. A metric space (X, d) has the *fixed point property* if and only if for any continuous function $f : X \to X$, f has a fixed point. [1]

2.2 Definition. Let (X, d_X) and (Y, d_Y) be metric spaces. X and Y are *homeomorphic* if and only if there exists a bijection $\boldsymbol{h} : X \to Y$ such that both h and h^{-1} are continuous.

Say that a property of sets/spaces is topological if and only if for any two homeomorphic spaces, *if one space has the property then so does the other*. Elsewhere, we have shown that compactness and connectedness are both topological properties. In P,convexity is topological because, in P, convexity is equivalent to connectedness. But, more generally, convexity is not topological.

For example, in P^2, a figure shaped like a five-pointed star is not convex even though it is homeomorphic to a convex set, a pentagon. On the other hand, the fixed point property is topological.

2.3 Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces. If X and Y are homeomorphic and X has the fixed point property then Y also has the fixed point property.

Proof. Suppose that X has the fixed point property, that h:X→Y is a homeomorphism, and that g:Y→Y is continuous. We need to show that g has a fixed point.

Define $f = h^{-1} \bullet g \bullet h$. Then $f : X \to X$ is continuous, as a composition of continuous functions. Since X has the fixed point property, there is a point $x^* \in X$ such that $f(x^*) = x^*$, meaning $h^{-1}(g(h(x^*)) = x^*$, or $g(h(x^*)) = h(x^*)$, which implies that $h(x^*)$ is a fixed point of g.

3. The Brouwer Fixed Point Theorem in P_{+}^{N+1}

3.1 Definition. Let the convex set Δ^N in P_{+}^{N+1} is called a regular N-simplex. Then

$$\Delta^N = \left\{ x = (x_1, \dots, x_{N+1}) \in P_{+}^{N+1} : \sum_{i=1}^{N+1} x_i = 1 \right\}.$$

More generally, a regular N-simplex is defined by N+1 equally spaced points, the vertices of the simplex. In the case of Δ^N the vertices are:

$e_1 = (1,0, \dots ,0)$, $e_2 = (0,1,0,\dots ,0)$, ... , $e_{N+1} = (0,\dots ,0, 1)$ all of which are distance 1 apart. A regular 1-simplex is a line segment (Fig. 1). A regular 2-simplex is an equilateral triangle (Fig. 2). A regular 3-simplex is a regular tetrahedron (Fig. 3).

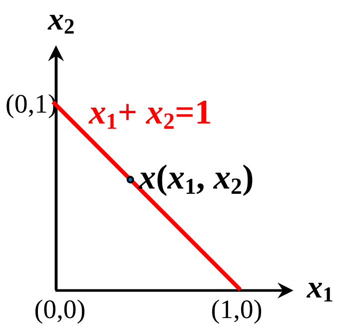

Fig. 1: A regular 1-simplex (a line)Δ^1 in P_{+}^{2}

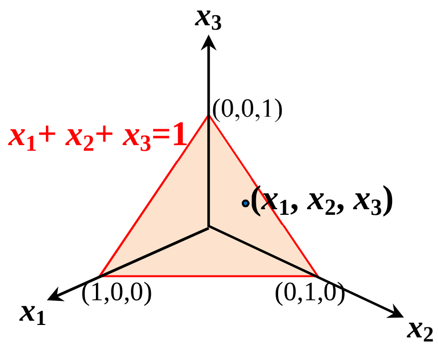

Fig. 2: A regular 2-simplex (equilateral triangle)Δ^2 in $\mathbf{P}_+^3$

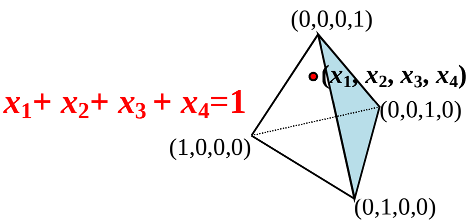

Fig. 3: A regular 3-simplex (tetrahedron)Δ^3in $\mathbf{P}_+^4$

Any regular N-simplex is (almost trivially) homeomorphic to Δ^N.

For each $t\in$ N, $t\geq1$, there is a *simplicial subdivision* of Δ^N, formed by introducing vertices at the points $\left(\frac{k_1}{t}, \dots, \frac{k_n}{t}, \dots, \frac{k_{N+1}}{t}\right)$ where $k_n\in$ N = { 0, 1, 2, ... } and $\sum_n k_n = t$. Note that this set of vertices is finite and includes the original vertices. The closest of these vertices are distance $\frac{1}{t}$ apart and any set of N+1 such closest vertices define an N-simplex that is a mini version ofΔ^N. Call these new simplexes sub-simplexes.

For example, if N = 2, (1, 0, 0), (0, 1, 0), (0, 0, 1)are vertices of Δ^2 .

If t = 2, then three new vertices are introduced, at $(\frac{1}{2},\frac{1}{2},0)$, $(\frac{1}{2},0,\frac{1}{2})$, $(0, \frac{1}{2}, \frac{1}{2})$ for a total of six vertices. These six vertices divide the original simplex into four sub-simplexes, each with sides of length $\frac{1}{2}$ .See Fig. 4.

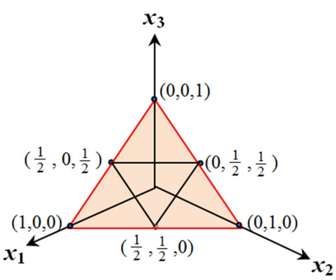

Fig. 4: Four sub-simplexes for t = 2

If t = 3, the simplex is divided into nine sub-simplexes, each with sides of length $\frac{1}{3}$.

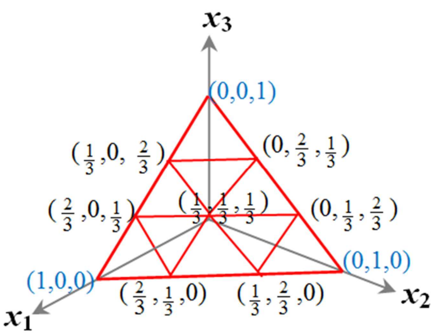

Fig. 5: Nine sub-simplexes for t = 3

3.2 Theorem. Δ^N has the fixed point property.

Proof: Fix a continuous function $f:\Delta^N \to \Delta^N$. If for any subdivision of Δ^N there is a fixed point at some vertex $v \in \Delta^N$ then we have done.

Suppose then that for every t, no vertex is a fixed point. By a labeling of a simplex we mean a function that assigns a number in $n \in \{1,\dots, N+1\}$ to each vertex. Say that a sub-simplex is completely labeled if and only if each of its N+1 vertices has a different label. The proof now proceeds in two steps.

(1) For each t consider any labeling such that if the label of vertex v is n then $f_n(v) < v_n$. Since $f(v) \neq v$ (by assumption, no vertex is a fixed point) and since $\sum_n f_n(v) = 1 = \sum_n v_n$, there *must be at least one* n for which $f_n(v) < v_n$, hence the labeling is well defined. (There may be more than one n for which $f_n(v) < v_n$, in such cases, any such n can be the label for v.)

Suppose that for each t, Δ^N has a at least one completely labeled sub-simplex.

Step Two shows that this is true. Let the N+1 vertices of this sub-simplex be $v_t^1, \dots, v_t^n, \dots, v_t^{N+1}$, where v_t^n has label n. The point $(v_t^1, \dots, v_t^n, \dots, v_t^{N+1})$ lies in $\Delta^N \times \Delta^N \times \dots \times \Delta^N$ (N+1 times), which is compact since Δ^N is compact. Therefore, there is a point $(x^{1*},\dots ,x^{(N+1)*})$ and a subsequence along which $(v_t^1, \dots, v_t^n, \dots, v_t^{N+1})$ converges to $(x^{1*},\dots ,x^{(N+1)*})$. For each t, the vertices of any sub-simplex are exactly $\frac{1}{t}$ apart. Therefore, for any $\varepsilon > 0$, for all t large enough, for any n, v_t^n is within ε of x^{1*}. This implies $x^{1*} = \dots = x^{(N+1)*}$. Call this common limit x^*.For each t and n, since the label on v_t^n is n, $f_n(v_t^n) < v_{tn}^n$:

Therefore, taking the limit, since f is continuous, $f_n(x^*) \le x_n^*$:

If any inequality is strict, then $\sum_n f_n(x^*) < \sum_n x_n^*$. But $f:\Delta^N \to \Delta^N$, hence $\sum_n f_n(x^*) < \sum_n x_n^* = 1$.

Hence $f_n(x^*) = x_n^*$ for all n: x^* is a fixed point of f, as was to be shown.

(2) Fix any Δ^N and any t simplicial subdivision. Consider any labeling of the vertices such that if the label of vertex v is n then $v_n > 0$. Note that this property was satisfied by the labeling in step (1). Otherwise, the labeling is unrestricted.

3.3 Theorem (Sperner's Lemma). Given Δ^N, a t simplicial subdivision, and a labeling as above, the number of completely labeled sub-simplexes is odd.

Proof: If $N = 0$ then $\Delta^N = 1$. The "simplex" is just the point $x = 1$, for any t, the simplicial subdivision is dim and the point is completely labeled (the label is 1). Consider now Δ^N, $N \geq 1$. This simplex has N+1 faces defined by any set of$N$ of the vertices. Each face is itself a copy of Δ^{N-1}.

Any t simplicial subdivision of the original Δ^N induces a t simplicial subdivision on each face. By the induction hypothesis, any face of Δ^{N+1}, being a copy of Δ^N, has an odd number of completely labeled (sub-) sub simplexes.

Given Δ^N, consider the N-dimensional plane containing Δ^N, namely

$$\{ x \in P^{N+1} : \sum_{n=1}^{N+1} x_n = 1 \}$$

For each t, one can construct a simplicial division of the plane that includes the t simplicial subdivision of the original Δ^N. This simplicial subdivision has vertices at $\left(\frac{k_1}{t}, \ \ldots \ , \frac{k_n}{t}, \ldots \ , \frac{k_{N+1}}{t}\right)$ for $k_n \in Z$, $\sum_{n=1}^{N+1} \frac{k_n}{t} = 1$.

For any face of the original Δ^N, consider any sub-simplex of this face. This sub-sub-simplex is a face shared by two sub-simplexes, one a sub-simplex of the original simplex and one that is not. Call this latter sub-simplex an *exterior sub-simplex*.

For any t, label Δ^N as above and consider the following sets.

– S_1: The set of completely labeled sub-simplexes of the original Δ^N.

– S_2: The set of sub-simplexes of the original Δ^N that have labels {1,... ,N} but that are missing label N+1 (and hence have one of the other labels repeated).

– S_3: The set of exterior sub-simplexes for which the face that is a (sub-)sub-simplex of the original Δ^N is completely labeled with labels {1,...,N} (not label N+1).

Let $S = S_1 \cup S_2 \cup S_3$. Let E be the set of sub-simplicial faces of Δ^N that are completely labeled with labels {1,...,N} (not label N+1). Note that for any $e \in E$, e is a face shared by two sub-simplexes in S. And any $s \in S$ has at least one face in E. For $s \in S$, let $deg(s)$ equal the number of faces in E. One can verify that, independently of N,

–For $s \in S_1 \cup S_3$, $deg(s) = 1$. – For $s \in S_2$, $deg(s) = 2$.

Then $\sum_{s \in S} deg(s) = 2\#E$,since each $e \in E$ is a face of two adjoining elements of S, and any $s \in S_1 \cup S_3$(which has degree 1) has only one face in E, while any $s \in S_2$(which has degree 2) has two faces in E.

This establishes that $\sum_{s \in S} deg(s)$ is even. On the other hand, $\sum_{s \in S} deg(s) = \#S_1 + 2\#S_2 + \#S_3$:

By the induction hypothesis, the number of completely labeled (sub-)sub-simplexeson any face of Δ^N is odd, hence $\#S_3$ is odd. Since $\sum_{s \in S} deg(s)$ is even, $2\#S_2$is even, and $\#S_3$ is odd, it follows that $\#S_1$ is odd, as was to be shown.

A natural conjecture is that the number of fixed points must, therefore, be odd, butthis is not true.

(1) For X = [0, 1], $f(x) = x$ has an infinite number of fixed points.

(2) For X = [0, 1], $f(x) = 6x^3 - 9x^2 + 5x - \frac{4}{9}$ has two fixed points. See Fig. 1.

These examples turn out to be pathological: in a sense that can be formalized, if X is homeomorphic to a compact, convex set then "nearly every" continuous function on X has an odd number of fixed points.

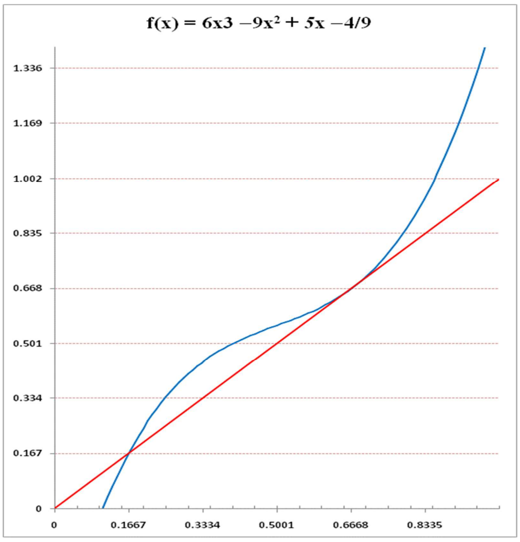

Fig. 1: $f(x) = 6x^3 - 9x^2 + 5x - \frac{4}{9}$ has two rational fixed points, $x = \frac{1}{6}$ and $x = \frac{2}{3}$.

A corollary of Theorem 3.2 is that any compact, convex subset of a Euclidean space has the fixed point property. This is the form in which Brouwer is typically stated.

3.4 Theorem (Brouwer Fixed Point Theorem).

If $A \subseteq P^N$ is compact and convex then it has the fixed point property.

Proof: Since A is bounded, it is a subset of a sufficiently large regular N-simplex, call it S, which in turn is homeomorphic to Δ^N.

Consider any continuous function $f: A \to A$. For each $x \in S$, consider the problem $min_{a\in A}$ $d(x,a)$. This problem has a solution, since A is compact and Euclidean distance is continuous. We claim that the solution is unique. Consider any solutions $a, b \in A$. Let γ note the minimum distance from A to x; then $d(x,a) = d(x,b) = \gamma$. We will show that $a = b$.

Let $c = \frac{a+b}{2}$, then $c \in A$, since A is convex, and so,

$$\gamma \le d(x,c) = \left\| x - \frac{a+b}{2} \right\| = \left\| \frac{x}{2} - \frac{a}{2} + \frac{x}{2} - \frac{b}{2} \right\| \le \left\| \frac{x}{2} - \frac{a}{2} \right\| + \left\| \frac{x}{2} - \frac{b}{2} \right\| = \frac{1}{2} d(x,a) + \frac{1}{2} d(x,b)$$

That implies $d(x, c) = \gamma$.

Thus, the second inequality above is an equality, which implies that $x-a$ and $x-b$ are positively collinear, which implies $a = b$. For each $x \in S$, let $\phi(x)$ be the unique solution to $min_{a\in A}$ $d(x,a)$.

We claim that $\phi: S \to A$ is continuous. This is a special case of the Theorem of the Maximum. To keep these notes somewhat self-contained, here is a proof. Consider any $\xi \in S$ and any sequence (x_t) in S such that $x_t \to \xi$. We need to show that $\phi(x_t) \to \phi(x)$. To simplify notation, let $a_t = \phi(x_t)$. Take any convergent subsequence (a_{tk}), converging to, say,α. (A convergent subsequence must exist since A is compact, but we actually do not need this fact at this point in the proof.) We claim that $\alpha = \phi(\xi)$ and that $a_t \to \alpha$.

Consider first any $x \in S$ such that $d(\xi, x) < d(\xi, \alpha)$. Then by continuity of d, for t_k large enough, $d(x_{tk}, x) < d(x_{tk}, a_{tk})$. Since $a_{tk} \in \phi(x_{tk})$, this implies $x \notin A$. By contraposition, if $a \in A$, then $d(\xi, a) < d(\xi, \alpha)$, hence $\alpha = \phi(\xi)$.Therefore, every convergent subsequence of (a_t) converges to the same point, namely$\alpha = \phi(\xi)$; since A is compact, this implies that $a_t \to \alpha$, hence $\phi(x_t) \to \phi(\xi)$.

Define $g : S \to A$ by $g = f \circ \phi$.

Note that for $a \in A$, $\phi(a) = a$, hence $g(a) = f(\phi(a)) = f(a)$. g is continuous since _ and f are continuous. Since S is homeomorphic to Δ^N, S has the fixed point property. Therefore, g has a fixed point,ξ, on S: $g(\xi) = \xi$. Since $g(x) \in A$ for every $x \in S$, $\xi \in A$. Since $g(a) = f(a)$for every $a \in A$, ξis a fixed point of f.

3.5 The Contraction Mapping Theorem

This section focuses on the Contraction Mapping Theorem, which places only an extremely weak restriction on the domain but imposes a very strong continuity condition on f.

To motivate the Contraction Mapping Theorem, consider first the case of an affine function on P: $f : P \to P$,$f(x) = ax + b$ where $a, b \in P$. If $a \neq 1$, then f has the fixed point $x^* = b/(1-a)$.

(If a = 1 and b = 0 then every point is a fixed point. If $a = 1$ and $b \neq 0$ then there is no fixed point.) Note that the domain of f here is not compact.

The following provides an algorithm for finding x^*. Of course, we already have a formula for x^*, so we don't need an algorithm. But the algorithm generalizes, where as the formula for x^* does not.

First we suppose that $|a| < 1$. Take x_0 to be any point in P.

Let $x_1 = f(x_0)$,$x_2 = f(x_1) = f(f(x_0))$, ... Then $x_t \to x^*$. This is easiest to see if $b = 0$, in which case $x^* = 0$. Then $x_1 = ax_0$, $x_2 = a^2x_0$, ... ,$x_t = a^t x_0$, ...

Since $|a| < 1$, $a^t \to 0$,which implies $a^t \to 0$, as was to be shown. If instead $|a| > 1$, then simply invert $y = ax + b$ to get,

$$x = f^{-1}(y) = \frac{1}{a} y - \frac{1}{b}.$$

Take y_0 to be any point in P, $y_1 = f^{-1}(y_0)$, and so on. Since $|a| > 1$, $\left|\frac{1}{a}\right| < 1$, and hence $y_t \to y^* = x^*$.

The Contraction Mapping Theorem extends this argument to non-linear functions in arbitrary complete metric spaces. In the case where the domain is P and the function is differentiable, the analog to the requirement in the affine case that $|a| < 1$ is that there is a number $c \in [0,1)$ such that for every $x \in P$, $|Df(x)| \le c < 1$.

For any $x,\xi \in P$, the Mean Value Theorem says that there is an $x_m \in (x,\xi)$ such that $Df(x_m) \approx \frac{f(\xi)-f(x)}{\xi - x}$ which implies $|f(\xi)-f(x)| \leq c|\xi - x|$.

This motivates the following definition.

3.6 Definition. Let (X, d) be a metric space. A function $f : X \to X$ is a *contraction*if and only if there is a number $c\in[0, 1)$ such that for any $\xi ,x\in X$,

$d(f(\xi),f(x)) \leq c\, d(\xi,x)$.

3.7 Theorem. (Contraction Mapping theorem).

Let (X, d) be a non-empty complete metric space. Then any contraction $f : X \to X$ has a unique fixed point.

Proof. Take any $x_0 \in X$ and form the sequence x_0, $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0))$, and so on. It is an easy to show that if f is a contraction then this sequence is Cauchy. Since X is complete, there is an $x^* \in X$ such that $x_t \to x^*$.

Since $x_{t+1} = f(x_t)$, and $x_t \to x^*$, this implies that $f(x_t) \to x^*$. Since f is a contraction,it is (trivially) continuous. Hence $f(x^*) = x^*$**,** which establishes that x^* is a fixed point. Finally, if x^* and ξ are both fixed points then, since f is a contraction, $d(f(\xi),f(x^*)) \leq c d(\xi,x^*) \Rightarrow d(\xi,x^*) \leq c d(\xi,x^*)$, which implies that $d(\xi,x^*) = 0$, which establishes that x^* is the unique fixed point.

3.8 Remark. If f is invertible and f^{-1} is a contraction then f^{-1}has a fixed point, and hence so does f.

Conclusion

Generalizations of the Brouwer theorem have appeared in relation to the theory of topological vector spaces in mathematical analysis. The compactness convexity, single-valuedness, continuity, self-mapness, and finite dimensionality related to the Brouwer theorem are all extended and moreover., for the case of infinite dimension, it is known that the domain and range of the map may have different topologies. This is why the Brouwer theorem has so many generalizations. Current study of its generalizations concentrates on a more general class of compact or condensing multimaps defined on convex subsets of more general topological vector spaces.

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