

A Stochastic Cellular Traffic Automaton Model

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Abstract

First we introduced a cellular traffic model with examples. Next we extended the stochastic cellular traffic automaton (SCTA) model for highway. Then we illustrated the trajectories of vehicles and observed the microscopic variables from the time-space diagram.

Keywords: cellular automaton, microscopic, velocity, flow, density, headway and spacing.

Notations: $[a]$ is the integer part of the real number a . " \leftarrow " means "takes the value of".

1. Cellular Traffic Model

The first step in constructing the model is to divide the road into equal segments, each with length ℓ as shown in Fig. 1. Usually, this distance is taken to be the length of an average car, or vehicle, on the road. Time is also divided into equal segments, producing a time step Δt . The objective of the model is, given the positions of the cars at time t_j , to determine their positions at $t_{j+1} = t_j + \Delta t$. With this in mind we introduce an integer variable m that equals the number of road segments the car moves in a time step Δt . Here $m \leq g$ where g is the number of empty segments (gap) between in front of the car.

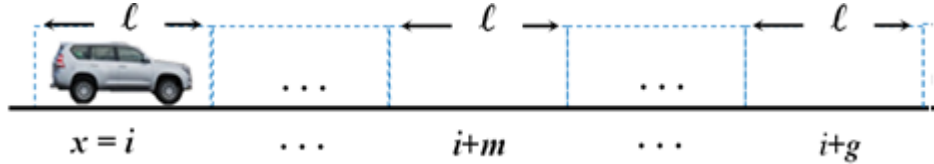


Fig. 1: Car moving diagram for cellular traffic model

It is assumed that there is a *maximum velocity* v_M that a car is allowed (on the roadway) to move in a *time step* Δt . Thus, the *maximum number of segments* allowed to move in time step Δt can be determined by $M = \lfloor \frac{\Delta t \times v_M}{\lambda} \rfloor$ segments. Thus, a car can move only m segments in a time step Δt where $m \leq \min\{M, g\}$. Then a car's *velocity* in this formulation is $v = \frac{m \ell}{\Delta t}$.

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Each car has three integers associated with $(m; x, g)_{t = \Delta t}$ where m is the number moving segments in time Δt , x is the current position of the car and g is the number of spaces in front of the car. Suppose that L is the total length of the current roadway then $g = 0, 1, 2, \dots, \lfloor L/\ell \rfloor - 1$.

1.1 Example. Given $\ell = 15$ ft, $\Delta t = 1$ sec, $v_M = 32$ mph (46.9 ft/s) and $L = 90$ ft. (See Fig. 2)

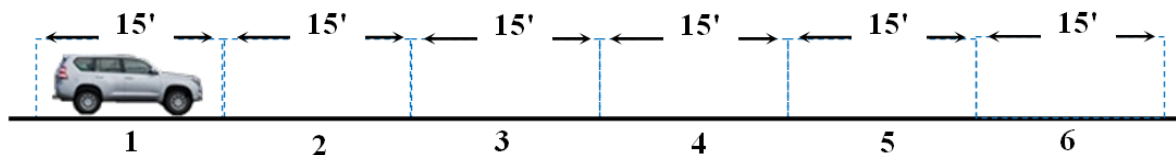


Fig. 2: Car following diagram for $L = 90$ ft

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The maximum segments allowed to move is $M = \left\lfloor \frac{\Delta t \times v_M}{\lambda} \right\rfloor = \left\lfloor \frac{1 \times 46.9}{15} \right\rfloor = \lfloor 3.13 \rfloor = 3$, number of empty spaces $g = \lfloor L/\lambda \rfloor - 1 = \lfloor 90/15 \rfloor - 1 = 6 - 1 = 5$ and $m \leq \min\{M, g\} = \min\{3, 5\} = 3$. Therefore the driver can choose only $m = 1$ or 2 or 3 to move forward.

By the definition, the initial position of a car at $t = 0$ sec can be written as $(0; 1, 5)_{t=0}$. The position for $t = 1$ sec can be written as follows:

If $m = 1$, we write $(1; 2, 4)_{t=1}$, if $m = 2$ then $(2; 3, 3)_{t=1}$ and if $m = 3$ then we write $(3; 4, 2)_{t=1}$.

The basic idea in this model is that at time t_0 we know the initial values of $(m_0; x_0, g_0)$ for each car, and what the model does is to determine the value $x_1 = x_0 + m_1$ where $m_1 \leq \min\{M, g_0\}$, $M = \left\lfloor \frac{\Delta t \times v_M}{\lambda} \right\rfloor$ and $g_1 = g_0 - m_1$. Here, $v_M \in \mathbb{R}^+$ is the maximum velocity that a car is allowed to drive on that roadway. Thus, we write the position of the car at time $t_1 = t_0 + \Delta t$ as $(m_1; x_1, g_1)$. Hence, the position of a car at time $t_j = t_0 + j\Delta t$ can be written as $(m_j; x_j, g_j)_{t=t_j}$; $j = 1, 2, \dots$ where

$$(1) \quad m_j = \text{rand}\{0, 1, \dots, M_j\} \text{ with } p = \frac{1}{M_j + 1}, M_j = \min\{M, g_{j-1}\},$$

$$(2) \quad x_j = x_0 + m_1 + \dots + m_j,$$

and

$$(3) \quad g_j = g_{j-1} - m_j.$$

1.2 Example. Given that $(0; 0, 15)$, $\Delta t = 1$ sec and $M = 3$.

Therefore, $m_1 = \text{rand}\{0, 1, \dots, \min\{3, 15\}\} = \text{rand}\{0, 1, 2, 3\}$, $x_1 = x_0 + m_1 = 0 + m_1$ and the gap $g_1 = g_0 - m_1 = 15 - x_1$. Suppose that 10 random numbers, $(m_j) = (1, 2, 3, 2, 1, 0, 1, 2, 1, 1)$, has been chosen then we write the movement (in the first 10 seconds) of the car as:

$$(1; 1, 14)_{t=1}, (2; 3, 12)_{t=2}, (3; 6, 9)_{t=3}, (2; 8, 7)_{t=4}, (1; 9, 6)_{t=5}, (0; 9, 6)_{t=6}, (1; 10, 5)_{t=7}, (2; 12, 3)_{t=8}, (1; 13, 2)_{t=9} \text{ and } (1; 14, 1)_{t=10}.$$

The total distance that the car travels in 10 sec is $14 \times 15 = 210$ ft, then the average speed may be 21 ft/s or 23 km/h. The time-space diagram for the car can be plotted as shown in Fig. 3.

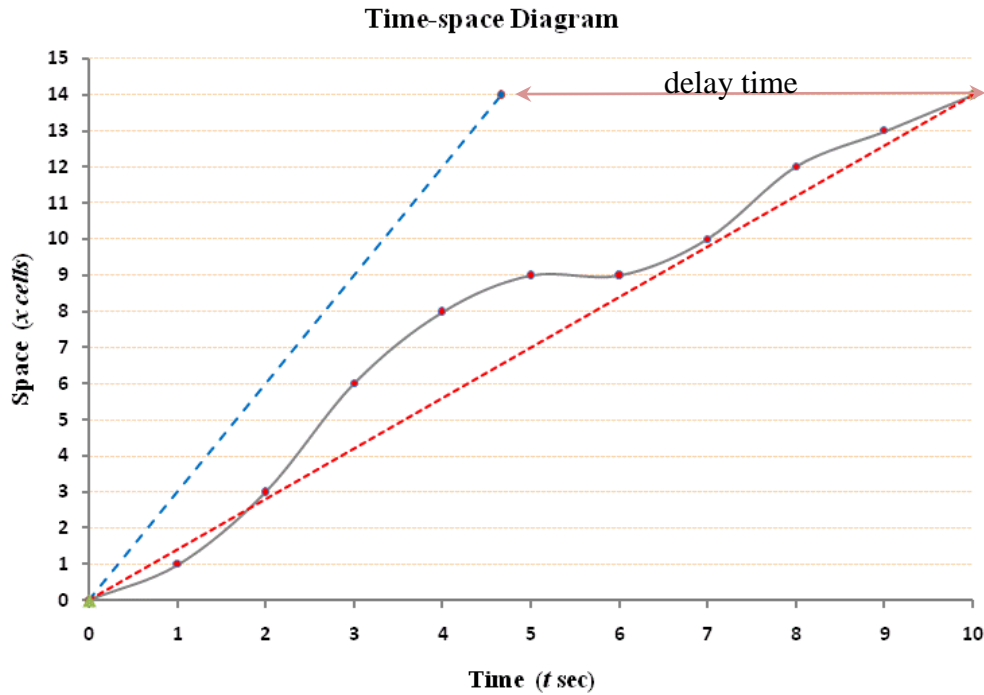


Fig. 3: Time-space diagram of a vehicle trajectory for $g_0 = 15, M_j = 3$ for $j = 1, 2, \dots, 10$.

The average speed v_{av} of the car can be calculated from the slope of the red dotted line, which is line segment joining (t_0, x_0) to (t_{10}, x_{10}) . Thus, $v_{av} = 14/10 = 1.4$ seg/s, that is 21 ft/s.

The minimum time to travel x_0 to x_{10} is $t_{min} = x_{10}/M_j = 14/3 = 4.6667$ sec. Thus, the free flow speed can be calculated from the slope of the blue dashed line, which is the line segment joining (t_0, x_0) to (t_{min}, x_{10}) . Thus, free flow velocity $v_{free} = 14/4.6667 = 3$ seg/s or 45 ft/s or 49.4 km/h.

Thus, the delay time = actual travel time – free flow time = $10 - 4.67 = 5.33$ sec.

2. Stochastic Cellular Traffic Automaton Model

It is difficult to determine exactly what happens in using analytical methods for a group of cars. The approach, therefore, is to use computer simulations and this brings us to the stochastic cellular traffic automaton (SCTA) model. To develop the SCTA model, we shall start with the following example.

2.1 Example. Suppose that the cars start out uniformly distributed along the highway with n segments, where the gap g is the same for all cars. Assume that $m \leq g \leq M \leq N$ for all cars.

See in Fig. 4(a), five cars are placed along a roadway at $t = 0$ with $N = 20, g = 3, M = 10$.

Then the positions of 5 cars are: $C_1 = (17, 3), C_2 = (13, 3), C_3 = (9, 3), C_4 = (5, 3)$ and $C_5 = (1, 3)$.

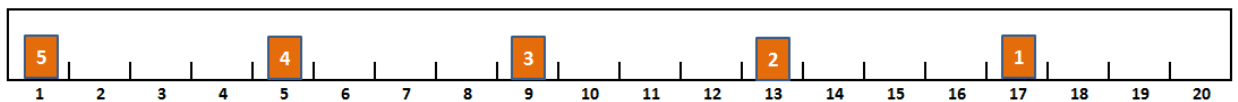


Fig. 4(a) $t = 0, M_j = g = 3$ for $j = 1, 2, 3, 4, 5$

For the first step ($t = 1$ sec):

By choosing $m_1 = \text{rand}\{0, 1, 2, M_1 = 3\}$ with $p = 1/4$, we get $m_1 = 1$.

Then $x_1 \leftarrow x_1 + m_1 = 17 + 1 = 18, g_1 \leftarrow M_1 - m_1 = 3 - 1 = 2$ and $C_1 = (x_1, g_1) = (18, 2)$.

By choosing $m_2 = \text{rand}\{0, 1, 2, 3, M_2 = 3 + 1 = 4\}$ with $p = 1/5$, we get $m_2 = 2$.

Then $x_2 \leftarrow x_2 + m_2 = 13 + 2 = 15, g_2 \leftarrow M_2 - m_2 = 4 - 2 = 2$ and $C_2 = (x_2, g_2) = (15, 2)$.

By choosing $m_3 = \text{rand}\{0, 1, 2, 3, 4, M_3 = 3 + 2 = 5\}$ with $p = 1/6$, we get $m_3 = 3$.

Then $x_3 \leftarrow x_3 + m_3 = 9 + 3 = 12, g_3 \leftarrow M_3 - m_3 = 5 - 3 = 2$ and $C_3 = (x_3, g_3) = (12, 2)$.

By choosing $m_4 = \text{rand}\{0, 1, 2, 3, 4, 5, M_4 = 3 + 3 = 6\}$ with $p = 1/7$, we get $m_4 = 4$.

Then $x_4 \leftarrow x_4 + m_4 = 5 + 4 = 9, g_4 \leftarrow M_4 - m_4 = 6 - 4 = 2$ and $C_4 = (x_4, g_4) = (9, 2)$.

By choosing $m_5 = \text{rand}\{0, 1, 2, 3, 4, 5, 6, M_5 = 3 + 4 = 7\}$ with $p = 1/8$, we get $m_5 = 4$.

Then $x_5 \leftarrow x_5 + m_5 = 1 + 4 = 5, g_5 \leftarrow M_5 - m_5 = 7 - 4 = 3$ and $C_5 = (x_5, g_5) = (5, 3)$.

Here $x_5 > g + 1 = 4$ then we shall insert a new car $C_6 = (1, x_5 - 2) = (1, 3)$ at the first place. Thus we have the Fig. 4(b).

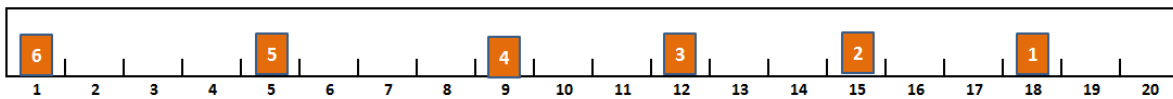


Fig. 4(b) the position of cars at $t = 1$ sec

For the second time step ($t = 2$ sec):

By choosing $m_1 = \text{rand}\{0, 1, M_1 = 2\}$ with $p = 1/3$, we get $m_1 = 1$.

Then $x_1 \leftarrow x_1 + m_1 = 18 + 1 = 19, g_1 \leftarrow M_1 - m_1 = 2 - 1 = 1$ and $C_1 = (x_1, g_1) = (19, 1)$.

If $m_2 = 2 \leq g_2 + m_1 = 3$ then $x_2 \leftarrow x_2 + m_2 = 15 + 2 = 17, g_2 \leftarrow M_2 - m_2 = 3 - 2 = 1$ and $C_2 = (17, 1)$.

If $m_3 = 2 \leq g_3 + m_2 = 4$ then $x_3 \leftarrow x_3 + m_3 = 12 + 2 = 14, g_3 \leftarrow M_2 - m_2 = 4 - 2 = 2, C_3 = (14, 2)$;

If $m_4 = 2 \leq g_4 + m_3 = 4$ then $x_4 \leftarrow x_4 + m_4 = 9 + 2 = 11, g_4 \leftarrow M_4 - m_4 = 4 - 2 = 2$ and $C_4 = (11, 2)$;

If $m_5 = 2 \leq g_5 + m_4 = 5$ then $x_5 \leftarrow x_5 + m_5 = 5 + 2 = 7, g_5 = 5 - 2 = 3$ and $C_5 = (7, 3)$;

If $m_6 = 2 \leq 5$ then $x_6 \leftarrow x_6 + m_6 = 1 + 2 = 3, g_6 = 5 - 2 = 3$ and $C_6 = (3, 3)$. Thus we have Fig. 4(c).

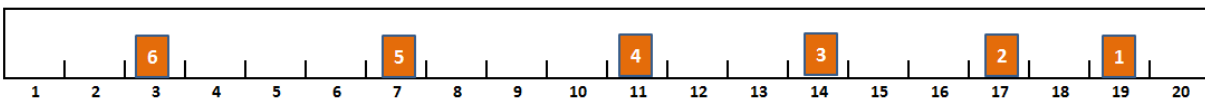


Fig. 4(c) the position of cars at $t = 2$ sec

For the third step ($t = 3$ sec):

if $m_1 = 2 > g_1 = 1$ then $x_1 \leftarrow x_1 + m_1 = 19 + 2 = 21 > N$. Thus, C_1 is removed from the diagram;

if $m_2 = 1 \leq 3$ then $g_2 = 3 - 1 = 2$ and $C_2 = (18, 2)$; if $m_3 = 2 \leq 3$ then $g_3 = 3 - 2 = 1, C_3 = (16, 1)$;

if $m_4 = 2 \leq 4$ then $g_4 = 4 - 2 = 2$ and $C_4 = (13, 2)$; if $m_5 = 2 \leq 5$ then $g_5 = 5 - 2 = 3, C_5 = (9, 3)$;

if $m_6 = 2 \leq 5$ then $g_6 = 5 - 2 = 3, C_6 = (5, 3)$.

Here we insert $C_7 = (1, 3)$ at the first place for $x_6 = 5 > g + 1$. Thus we have the Fig. 4(d).

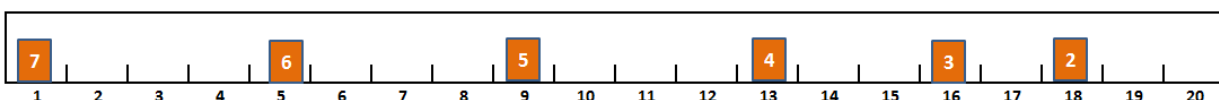


Fig. 4(d) the position of cars at $t = 3$ sec

Fig. 4: Diagram of car positions along the roadway for 3 time steps

Now we have to define a rule for the appearance of a new car on the left and disappearance of the first car on the right.

- (i) If the first cell is empty and the position of the last car is greater than $g+1$ then we will add a new car at the first position.
- (ii) If the position of the first car is greater than N , the first car is removed from the roadway.

The time-space diagram for the above example is as shown in Fig. 5.

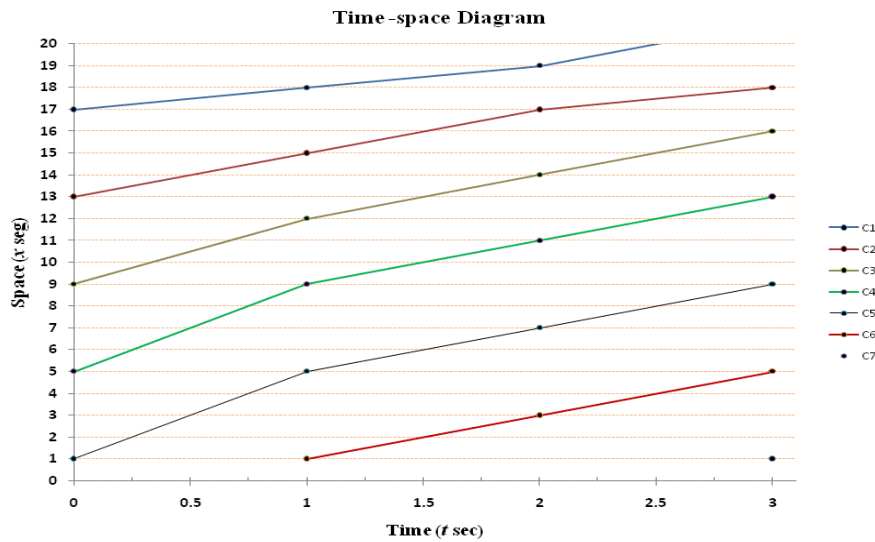


Fig. 5 Time-space diagram of traffic flow for example 2.1

2.2 Trajectories and Microscopic variables

To determine the traffic variables we now present a time-space diagram for two vehicles.

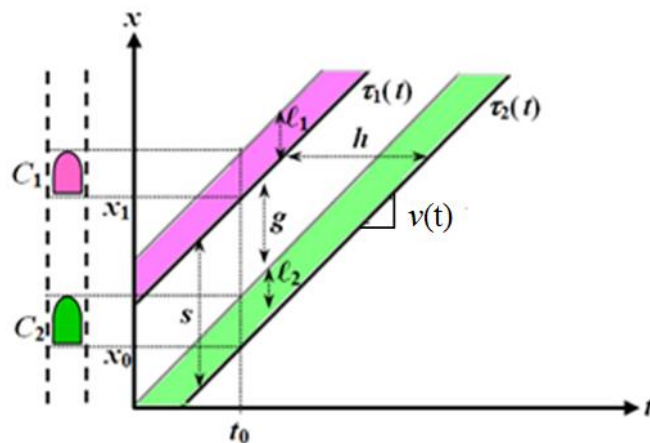


Fig. 6 Time-space diagram for two vehicles C_1 and C_2

Let $\tau_1(t)$ and $\tau_2(t)$ be trajectories of the vehicles C_1 and C_2 , respectively. From Fig. 6, we can observe the following traffic parameters:

(1) Headway h : time between two vehicles passing through a point.

Let C_1 and C_2 passing through the point $x = x_1$ at time t_0 and t_1 , respectively. Then headway $h = t_1 - t_0$ (time unit).

(2) Spacing s : front to front (back to back) distance at a given time.

At $t = t_0$, $s = x_1 - x_0$ (length unit). $s = g + \lambda_2$ (See Fig. 6).

(3) Flow q : number observed at a point divided by time interval. Let n be number of cars pass through a point in time interval Δt .

Then the traffic flow $q = n/\Delta t$ veh/(time unit).

(4) Density k : number observed on a roadway at a given time divided by the total length.

Let n be number of vehicles in a time on a roadway of length L and N be the maximum number of vehicles queue in the length L .

Then, $k = n/L$ veh/length and maximum density $k_{jam} = N/L$ veh/length, $n \leq N$.

(5) Velocity v : The velocity of vehicle C_2 is the slope of the line $\tau_2(t)$, $v(t) = \Delta x/\Delta t$ (speed unit). We can prove $v(k) = v_{max}(1 - k/k_{jam})$ where v_{max} is the maximum speed allowed on the roadway and k_{jam} is the jam density.

(6) Fundamental relation: $q = v \times k$. Since $v(k) = v_{max}(1 - k/k_{jam})$ then $q(k) = v_{max}(1 - k/k_{jam})k$.

It is possible to write the computer code that produced Fig. 4 and calculate car positions using a large number of road segments and time steps. It is better to study specific situations and compare the results with what is expected on a real roadway.

References

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- [2] Rajeswaran, S., and Rajasekaran, S. (2013). "A Study of Vehicular Traffic Flow Modeling Based on Modified Cellular Automata", IOSR Journal of Mathematics ISSN: 2278-5728. Volume 4, Issue 5, 32-38.