Complete Bipartite Graphs and Their Line Graphs

KhinSandar Win*

Abstract

It is well-known that the line graph L(G) of a graph G(V, E) is a graph whose vertex set is E and two vertices of L(G) are adjacent if their corresponding edges are adjacent in G. In this paper, we investigate two generalizations of the concept of the classical line graphs which are known as the P_3-graphs and the line graphs of order k where k is a nonnegative integer. Mainly we find out some fundamental properties and structures of P_3-graphs and the line graphs of order 1 of complete bipartite graphs.

Keywords: line graph, P_3- graph, line graph of order k, complete bipartite graph

1. Preliminaries

A *graph G* is an ordered triple ($V(G)$, $E(G)$, ψ_G) consisting of a nonempty set $V(G)$ *ofvertices*, a set $E(G)$, disjoint from $V(G)$, of *edges*, and an *incidence function* ψ_G that associates with each edge of G and unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to *join u* and v or that the vertices u and v are called the *ends* of e, we also say that u and v are *incident* with e or that u and v are *adjacent*. Each vertex is indicated by a point, and each edge by a line joining the points which represent its ends. An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. The edges with the same end vertices are called the *multiple edges*.

A graph is called a *simple graph* if it has no loops and no multiple edges. A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. The complete graph with n vertices is denoted by K_n.

A*bipartite graph*is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y, such partition (X, Y) is called a *bipartition* of the graph. A *complete bipartite graph*is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $| X | = m$ and $| Y | = n$, such a graph is denoted by $K_{m,n}$. The *degree*of a vertex v_i in a graph G, $d_G(v_i) = d_i$, is the number of edges which are incident with v_i where each loop counting as two edges.

Let G = (V, E) be a graph with the vertex set V and edge set E. A *walk* in G is a finite non-null sequence W = $v_0e_1v_1e_2v_2 \ldots e_kv_k$whoseterms arealternately vertices and edges such that for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. If it does not lead to confusion, we will simply denote the walk W by the sequence of vertices $v_0v_1v_2 \ldots v_k$. The vertices $v_0$andv_k are called the *origin* and *terminus* of W respectively and $v_0v_1v_2 \ldots v_k$ are its *internal vertices*. The positive integer k, the number of edges in W, is the *length* of W. A walk is called a *path* if there are no vertex repetitions, a path with k vertices will be denoted by P_k. A *cycle* is a path in which its origin and terminus are the same, a cyclewith k vertices will be denoted by C_k.

Two vertices u and v of G are said to be *connected* if there is a uv-path in G. If there is a uv-path in G for any two distinct vertices u and v of G, then G is said to be *connected*, otherwise G is *disconnected*.

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* Dr., Professor, Department of Mathematics, West Yangon University

2. Line Graphs of Complete Bipartite Graphs

The concept of the line graph of a graph has provided a framework for the study of various structural properties of P_3-graphs and line graphs of order k. So we observe some interesting characterizations of the line graphs of complete bipartite graphs in this section.

2.1 Definition. Let G be a graph. The *line graph L(G)* of a graph G is a graph whose vertices are edges of G and joining two vertices whenever the corresponding edges in G are adjacent.

After defining the line graph of a graph, we state the following theorem which is studied by Moon [1963] and Hoffman [1964].

2.2 Theorem. If G is the line graph $L(K_{m,n})$ of a complete bipartite graph $K_{m,n}$, $m \geq 1$, $n \geq 1$, then

(i) G has mn vertices.

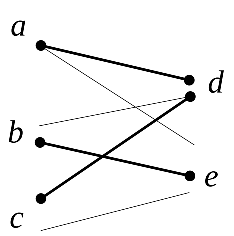
(ii) G is $(m + n - 2)$-regular,

(iii)every two nonadjacent vertices in G are mutually adjacent to exactly two vertices, and

(iv)among the adjacent pairs of vertices in G, exactly $n\binom{m}{2}$ pairs are mutually adjacent to exactly $m - 2$ vertices, and the other $m\binom{n}{2}$ pairs to $n - 2$ vertices.

2.3 Example. Fig. 2.1 shows a complete bipartite graph $K_{3,2}$ and its line graph $L(K_{3,2})$.

$K_{3,2}$:



$L(K_{3,2})$:

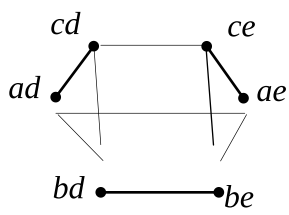


Fig. 2.1

In this figure we can easily see that

(i) $L(K_{3,2})$ has $3 \times 2 = 6$ vertices,

(ii) $L(K_{3,2})$ is $3 + 2 - 2 = 3$-regular,

(iii)two nonadjacent vertices *ad* and *ce* in $L(K_{3,2})$ are mutually adjacent to exactly two verticesae and *cd* , and (iv)among the adjacent pairs of vertices in $L(K_{3,2})$, exactly $2\binom{3}{2} = 6$ pairs (namely, (*ad* ,*bd*), (*ad* ,*cd*),(*bd* ,*cd*),(*ae* ,*be*),(*ae* ,*ce*), and (*be* ,*ce*)) are mutually adjacent to exactly 3 – 2 = 1 vertex, and the other $3\binom{2}{2} = 3$ pairs (namely, (*ad* ,*ae*), (*bd* ,*be*), and (*cd* ,*ce*)) to no vertex (i. e., 2 – 2 = 0).

3. P_3-graphs of Complete Bipartite Graphs

In this section we examine some properties of the socalled P_3-graphs which are introduced by Broersma and Hoede [1989] as a natural extension of the classical line graphs. Next we discuss some results on P_3-graphs of complete bipartite graphs which are the observations of ours.

3.1 Definition. Let G be a connected graph with at least two edges. The *P_3-graph $P_3(G)$* of G is the graph whose vertices are paths P_3 (i.e., paths with three vertices) in G and joining two vertices whenever the union of the corresponding paths P_3 in G forms a path P_4 or a cycle C_3.

The following theorems of Broersma and Hoede [1989] give the number of vertices and edges in a P_3-graph of a graph and the degree of a vertex in it.

3.2 Theorem. Let G = (V, E)be a graph, $P_3(G) = (V', E')$ its P_3-graph, and let I(G) be the set of vertices of G with degree greater than 1. Then

$$|V'| = \sum_{v \in I(G)} \binom{d_G(v)}{2}$$

$$|E'| = \frac{1}{2}\sum_{v \in V}\left[(d_G(v)-1)\sum_{u \in N(v)}(d_G(u)-1)\right]$$

where N(*v*) denotes the set of neighbouring vertices of *v* in G.

3.3 Theorem. Let G be a graph, $P_3(G)$ its P_3-graph and let $x = vmw$be a vertex in $P_3(G)$ where *v*, *m* and *w* are vertices in G. Then $d_{P_3(G)}(x) = d_G(v) + d_G(w) - 2$.

The following results are the observations of ours.

3.4 Theorem. Let $K_{m,n}$, $m \geq 1$, $n \geq 1$, be a complete bipartite graph. Then

(i) for all $n \geq 2$, $P_3(K_{1,n}) = \binom{n}{2} K_1$, (ii)for all $n \geq 2$, $P_3(K_{n,n})$ is a $(2n-2)$-regular bipartite graph with $2n\binom{n}{2}$ vertices and with a bipartition (X, Y) where $|X| = |Y| = n\binom{n}{2}$, and

(iii)for all m, n with $m \neq n$, $m \geq 2$, $n \geq 2$, $P_3(K_{m,n})$ is a nonregular bipartite graph with $m\binom{n}{2} + n\binom{m}{2}$ vertices and with a bipartition (X, Y) where $|X| = m\binom{n}{2}$, $|Y| = n\binom{m}{2}$, each vertex in X is of degree $2(m-1)$ and each vertex in Y is of degree $2(n-1)$.Proof.(i) By the structure of $K_{1,n}$, it contains exactly $\binom{n}{2}$ paths with three vertices, these

paths have a common middle vertex and the union of any two of them is neither a P_4 nor a C_3. Therefore $P_3(K_{1,n})$ has $\binom{n}{2}$ vertices and no edges and hence it is $\binom{n}{2}K_1$.

(ii) Let (A, B) with $A = \{u_1, u_2, \ldots, u_n\}$and $B = \{v_1, v_2, \ldots, v_n\}$ be a bipartition of $K_{n,n}$. Consider the P_3-graph $P_3(K_{n,n})$ of $K_{n,n}$. By Theorems 3.2 and 3.3, there are $2n\binom{n}{2}$ vertices in $P_3(K_{n,n})$ and each vertex is of degree $2n-2$. So $P_3(K_{n,n})$ is a $(2n-2)$-regular graph. Moreover the vertex set $V(P_3(K_{n,n}))$ of $P_3(K_{n,n})$ can be partitioned into two sets such that

$X = \{x \in V(P_3(K_{n,n})) \mid$ x corresponds to a path in $K_{n,n}$ containing three vertices and having an element of A as a middle vertex of that path$\}$,

$Y = \{y \in V(P_3(K_{n,n})) \mid$ y corresponds to a path in $K_{n,n}$ containing three vertices and having an element of B as a middle vertex of that path$\}$.

Then it is easy to see that no two vertices of X and no two vertices of Y are adjacent in $P_3(K_{n,n})$. Therefore $P_3(K_{n,n})$ is a bipartite graph with a bipartition (X, Y) where

$|X| = |Y| = n\binom{n}{2}$.

(iii)Let (A, B) with $A = \{u_1, u_2, \ldots, u_m\}$and $B = \{v_1, v_2, \ldots, v_n\}$ be a bipartition of $K_{m,n}$. Consider the P_3-graph $P_3(K_{m,n})$ of $K_{m,n}$. By Theorems 3.3, there are $m\binom{n}{2} + n\binom{m}{2}$ vertices in $P_3(K_{m,n})$. In this sum the first term represents the number of vertices that correspond to the paths in $K_{m,n}$ containing three vertices and having an element of A as a middle vertex, and the second term represents the number of vertices that correspond to the paths in $K_{m,n}$ containing three vertices and having an element of B as a middle vertex. Let

$X = \{x \in V(P_3(K_{m,n})) \mid$ x corresponds to a path in $K_{m,n}$ containing three vertices and having an element of A as a middle vertex },

Y = {y $\in$V(P_3($K_{m,n}$)) | y corresponds to a path in $K_{m,n}$ containing three vertices and having an element of B as a middle vertex }.

Then (X, Y) is a bipartition of the vertex set of P_3($K_{m,n}$) with $|X| = m\binom{n}{2}$ and

$|Y| = n\binom{m}{2}$ .It is also easy to see that no two vertices of X and no two vertices of Y are adjacent in P_3($K_{m,n}$). Therefore P_3($K_{m,n}$) is a bipartite graph with a bipartition (X, Y). By Theorem 3.3, each vertex in X is of degree m + m – 2 = 2(m – 1)and each vertex in Y is of degree n + n – 2 = 2(n – 1). Thus P_3($K_{m,n}$) is a nonregular graph.□

3.5 Example. Fig.3.1 shows complete bipartite graphs $K_{1,3}$, $K_{2,2}$, $K_{3,2}$and their P_3- graphs.

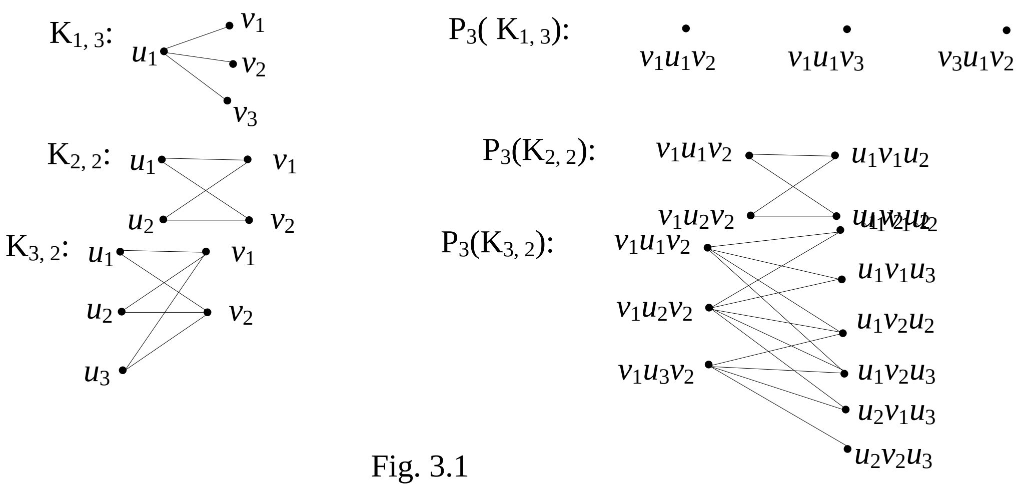


Fig. 3.1

4. Line Graphs of Order 1 of Complete Bipartite Graphs

One of the extension of line graph of a given graph, the line graph of order k of a given graph, is introduced in this section. Then we investigate line graphs of order 1 of complete bipartite graphs.

4.1 Definition. Let G be a simple connected graph containing at least one edge. The *line graph$L_k(G)$ of order k* of G is a graph whose vertices are edges of G and joining two vertices whenever the distance between the corresponding edges in G is k (k $\geq$ 0). By the *distance between two edges* $e_1 = uv$ (joining two vertices *u* and *v*) and $e_2 = xy$ (joining two vertices *x* and *y*), we mean the length of a shortest path in G joining a vertex of the set {*u*, *v*} and a vertex of the set {*x*, *y*}.

4.2 Example. A graph G and its line graphs of order 0, 1, 2are shown in Fig. 4.1.

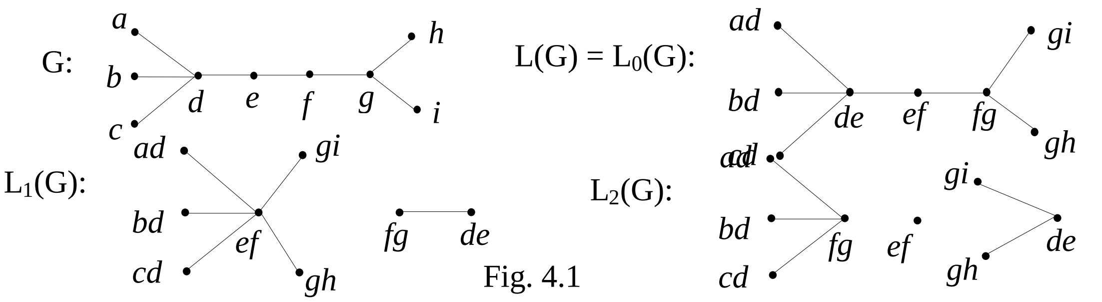


Fig. 4.1

With respect to the structures of the line graphs of order 1 of complete bipartite graphs: we have the following results

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4.3 Theorem. Let $K_{m,\,n}$, $m \geq 1$, $n \geq 1$, be a complete bipartite graph. Then$L_1(K_{m,\,n})$ is $(m - 1)(n - 1)$-regular.

Proof. We have $|V(L_1(K_{m,\,n}))| = |E(K_{m,n})| = mn$. Consider a vertex $x = uv$ in $L_1(K_{m,\,n})$ where u and v are vertices in $K_{m,\,n}$ incident with x. Let

$$A = \{\ a \in V(L_1(K_{m,n})|\ a \text{ corresponds to an edge of } K_{m,\,n} \text{incident with } u \text{or} v \text{ or both}\}.$$

Obviously, $|A| = m + n - 1$ and each vertex $a \in A$ is not adjacent to x. On the other hand for any vertex $b \in V(L_1(K_{m,\,n})) \setminus A$ corresponds to an edge, say cd of $K_{m,\,n}$, the distance between uv and $cd$is 1 since $K_{m,\,n}$ is complete bipartite. Thus every vertex $b \in V(L_1(K_{m,\,n})) \setminus A$ is adjacent to x. Therefore $L_1(K_{m,\,n})$ is $(m - 1)(n - 1)$-regular since

$$d_{L_1(K_{m,n})}(x) = \left|V(L_1(K_{m,n})) \setminus A\right| = \left|V(L_1(K_{m,n}))\right| - |A| = mn - (m+n-1) = \ldots = (m-1)(n-1). \ \square$$

4.4 Example. Fig. 4.2shows the complete bipartite graph $K_{2,\,4}$and its line graphs of order 1. It is interesting to observe that $L_1(K_{2,\,4})$ is a 3-cube.

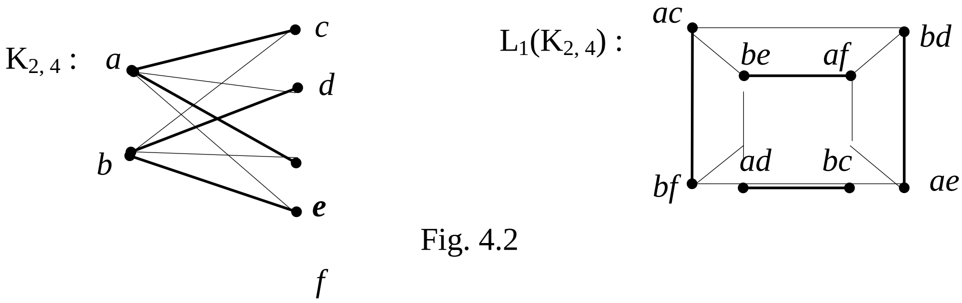

Fig. 4.2

The following theorem which describes this result can be seen as a counterpart of Theorem 2.2 studied on classical line graphs of complete bipartite graph.

4.5 Theorem. Let $K_{m,\,n}$, $m \geq 2$, $n \geq 2$, be a complete bipartite graph with a bipartition (X, Y) where $X = \{u_1, u_2, \ldots , u_m\}$and $Y = \{v_1, v_2, \ldots , v_n\}$. Then in $L_1(K_{m,\,n})$, (i)every two adjacent vertices are mutually adjacent to exactly (m – 2)(n −2) vertices, (ii)among the nonadjacent pairs of vertices, exactly $m\binom{n}{2}$ pairs are mutually adjacent to exactly (m – 1)(n – 2) vertices, and the other $n\binom{m}{2}$ pairs to (m – 2)(n −1) vertices.

Proof.(i) Consider the two adjacent vertices *a* and *b* in $L_1(K_{m,n})$. Since $K_{m,n}$ is complete bipartite, it is easy to see that *a* and *b* correspond to two nonadjacent edges in $K_{m,n}$. So we can assume that $a = u_iv_j$ and $b = u_kv_l$ where $u_i, u_k \in X$ and $v_j, v_l \in Y$. let

$A = \{ c \in V(L_1(K_{m,n})| \ c$ corresponds to an edge of $K_{m,n}$ incident with two vertices of $V(K_{m,n}) \setminus \{u_i, u_k, v_j, v_l\}\}$.

Then $|A| = |E(K_{m-2,n-2})| = (m-2)(n-2)$, *a* and *b* are mutually adjacent to each of (m − 2)(n − 2) vertices in A because $K_{m,n}$ is a complete bipartite graph. On the other hand each vertex $d \in V(L_1(K_{m,n}) \setminus A$ corresponds to an edge of $K_{m,n}$ incident with at least one of u_i, u_k, v_j, v_l and it follows that *a* and *b* are not mutually adjacent to each vertex of $V(L_1(K_{m,n})) \setminus A$. Therefore the number of vertices in $L_1(K_{m,n})$ to which *a* and *b* are mutually adjacent is (m − 2)(n − 2).

(ii) Consider the two nonadjacent vertices *p* and *q* in $L_1(K_{m,n})$. Since $K_{m,n}$ is complete bipartite, *p* and *q* correspond to two adjacent edges of $K_{m,n}$.

Case (i). Let $p = u_iv_j$ and $q = u_iv_l$ where $u_i \in X$ and $v_j, v_l \in Y$. We observe that there are $m\binom{n}{2}$ pairs of such nonadjacent vertices in $L_1(K_{m,n})$. This is because u_i can occur in m ways and the pair v_j, v_l can occur in $\binom{n}{2}$ ways. Let

$B = \{ r \in V(L_1(K_{m,n}))| \ r$ corresponds to an edge of $K_{m,n}$ incident with two vertices of $V(K_{m,n}) \setminus \{u_i, v_j, v_l\}\}$.

Then $|B| = |E(K_{m-1,n-2})| = (m-1)(n-2)$, *p* and *q* are mutually adjacent to each of (m − 1)(n − 2) vertices in B because $K_{m,n}$ is a complete bipartite graph. On the other hand each vertex *s* $V(L_1(K_{m,n})) \setminus B$ corresponds to an edge of $K_{m,n}$ incident with at least one u_i, v_j, v_l and it follows that *p* and *q* are not mutually adjacent to each vertex of $V(L_1(K_{m,n})) \setminus B$. Therefore the number of vertices in $L_1(K_{m,n})$ to which *p* and *q* are mutually adjacent is (m − 1)(n − 2).

Case (ii). Let $p = u_iv_j$ and $q = u_kv_j$ where $u_i, u_k \in X$ and $v_j \in Y$. We observe that there are $n\binom{m}{2}$ pairs of such nonadjacent vertices in $L_1(K_{m,n})$. This is because v_j can occur in n ways and the pair u_i, u_k can occur in $\binom{m}{2}$ ways. Let

$C = \{ r \in V(L_1(K_{m,n}))| \ r$ corresponds to an edge of $K_{m,n}$ incident with two vertices of $V(K_{m,n}) \setminus \{u_i, u_k, v_j\}\}$.

Then $|C| = |E(K_{m-2,n-1})| = (m-2)(n-1)$, *p* and *q* are mutually adjacent to each of (m − 2)(n − 1) vertices in C because $K_{m,n}$ is a complete bipartite graph. On the other hand each vertex $s \in V(L_1(K_{m, n})) \setminus C$ corresponds to an edge of $K_{m, n}$ incident with at least one u_i, u_k, v_j and it follows that p and q are not mutually adjacent to each vertex of $V(L_1(K_{m, n})) \setminus C$. Therefore the number of vertices in $L_1(K_{m, n})$ to whichn $\binom{m}{2}$ pairs of p and q are mutually adjacent is $(m-2)(n-1)$. □

Conclusion

The following table summarize the result on the line graph of order 1 of complete bipartite graphs and also compare it with the corresponding results on the classical line graph and P_3-graph.

Acknowledgements

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References

Graphs G	Line Graphs L(G)	P_3-Graphs $P_3(G)$	Line Graphs of Order 1 $L_1(G)$
Complete Bipartite Graph $K_{m, n}$	$(m + n - 2)$-regular graph for $m \geq 1, n \geq 1$	$\binom{n}{2} K_1$ for $m = 1, n \geq 2$ $(2n - 2)$-regular bipartite graph for $m = n \geq 2$ nonregular bipartite graph for $m \neq n$, $m \geq 2, n \geq 2$	nK_1 for $m = 1, n \geq 1$ $2K_2$ for $m = n = 2$ $(m - 1)(n - 1)$-regular graph for $m \geq 2, n \geq 2$

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