A Survey on b- metric, Pseudo metric and Hausdorff metric

Moe Moe Min Tun[1]

Abstract

The purpose of this research paper is to make a survey on b-metric, pseudo-metric and Hausdorff metric. Relations among them are also mentioned in this paper. Especially, we prove that Hausdorff metric is a metric.

1. Introduction

Nowsaday Nowadays, metric space is extended by inventing b-metric on a non-empty set. Hence first part of the theory will be extended and we many have new results relating to b-metric spaces.

There are a number of finite ways to define different kinds of metrics. Metric is extended by inventing b-metric. Pseudo-metric and Hausdorff metric are b- metric.

There are four parts in the organization of this research paper. The first part concerns b-metric. The second part relates pseudo metric. The third involves examples on b-metric and the last part concernsHausdorff metric.Relation among them are invested

2. b-metric

In this part, we study how to define b-metric on a nonempty set X of real numbers.

2.1 Definition: Let X be a non empty set and $k \geq 1$ be a given constant. A function $d: X \times X \rightarrow R^+$ is called a b-metric if the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, y) \leq k\,[d(x, z) + d(z, y)]$ for all $x, y, z \in X$

The pair (X, d) is called a b-metric space (with constant k). It is easy to see that any metric space is a b-metric space with k =1. The following example shows that a b-metric on X need not be a metric on X.

2.2 Theorem Every b-metric is a metric on X.

Proof: Let $d: X \times X \rightarrow R^+$ be above b-metric on X.

Then $d(x, y) \leq k\,[d(x, z) + d(z, y)]$ for all $x, y, z \in X$ for $k \geq 1$ by (iii).

When k = 1,

$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$

by (i) and (ii)

So d is a metric on X.

[1] Lecturer, Department of Mathematic, East Yangon University

Remark: Converse of Theorem 2.2 is false see Example 2.5.

3. Pseudo-metric

In this part, we study how to define pseudo-metric on a non-empty set X of real numbers.

3.1 Definition: Let d : X × X → R

$(x, y) \mapsto d(x, y) = \big|\, |x| - |y| \,\big|$. Then d is a pseudo - metric on X.

3.2 Theorem. Every pseudo - metric is a b-metric on X.

Proof: Clearly $d(x, y) \geq 0$ for all x, y ϵ X and

$d(x, y) = d(y, x)$ for all x, y ϵ X

$$\begin{aligned} d(x, y) &= ||x| - |y|| \\ &= ||x| - |z| + |z| - |y|| \\ &\leq ||x| - |z|| + ||z| - |y|| \\ &= d(x, z) + d(z, y) \end{aligned}$$

for all x, y, z ϵ X, here k = 1.

So d is a b-metric.

3.3 Example. The set R of real numbers together with the function

$d(x, y) = |x - y|^2$ for all x, y ϵ R is a b-metric space with constant k=2 but not a metric space on R.

Proof: Let d: R × R → R

$(x, y) \mapsto d(x, y) = |x - y|^2$.

Then d is a b-metric on R with k =2.

Since $|x - y|^2 \geq 0$ for all x, y ϵ R, $d(x, y) \geq 0$

$$\begin{aligned} |x - y| &= |y - x| \\ d(x, y) &= d(y, x) \text{ for all } x, y \in R \\ &= |x - y|^2 \\ &= |x - z + z - y|^2 \\ &\leq (|x - z| + |z - y|)^2 \\ &= |x - z|^2 + |z - y|^2 + 2|x - z|\,|z - y| \\ d(x, y) &\leq |x - z|^2 + |z - y|^2 + |x - z|^2 + |z - y|^2 \\ &= 2(|x - z|^2 + |z - y|^2) \\ &= 2(d(x, z) + d(z, y)) \text{ for all } x, y, z \in R \end{aligned}$$

So d is a b-metric on R.

Choose $x = -1$, y = 1, z = 0

d (x, y) $=|-1-1|^2=4$

d (x, z) $=|-1-0|^2=1$

d (z, y) $=|0-1|^2=1$

d (x, y) □d (x, z) + d (z,y)

So d is not a metric onR.

3.4 Example Let d: X × X → R

$(x, y) \mapsto d(x,y) = \sqrt{|x-y|}$. Then d is a metric on X.

Proof: d (x, y) ≥ 0 and d(x, y) = d(y, x) are obvious.

We will prove that

d (x, y) ≤ d (y, z) + d(z, y) for all x, y, z ϵ X

Since $|x-y| \le |x-z| + |z-y|$,

$|x-y| \le |x-z| + |z-y| + 2\sqrt{|x-z|\,|z-y|}$holds $|x-z|, |z-y| \ge 0$

Taking square root,

$$\sqrt{|x-y|} \le \sqrt{\left(\sqrt{|x-z|} + \sqrt{|z-y|}\right)^2}$$

$\sqrt{|x-y|} \le \sqrt{|x-z|} + \sqrt{|z-y|}$for all x, y, z ϵ R

So, d is a metric on X.

So, d is a b-metric on X.

3.5 Example Let d: X × X → R

$(x, y) \mapsto$ d (x, y) = $\sqrt[3]{|x-y|}$. Then d is a b-metric on X.

Proof: d (x, y) ≥ 0 and d (x, y) = d (y, x) are obvious for all x, y ϵ X.

We will prove d (x, y) ≤ d (x, z) + d (z, y) for all x, y, z ϵ X.

Since $|x-y| \le |x-z| + |z-y|$ and

$$\left(\sqrt[3]{|x-z|} + \sqrt[3]{|z-y|}\right)\sqrt[3]{|x-z|}\ \sqrt[3]{|z-y|} \ge 0$$

$$|x-y| \le |x-z| + |z-y| + 3\left(\sqrt[3]{|x-z|} + \sqrt[3]{|z-y|}\right)\sqrt[3]{|x-z|}\ \sqrt[3]{|z-y|}$$

$$=\left(\sqrt[3]{|x-z|} + \sqrt[3]{|z-y|}\right)^3$$

Taking cube root,

$\sqrt[3]{|x-y|} \le \sqrt[3]{|x-z|} + \sqrt[3]{|z-y|}$ for any x, y, z ϵ X

d(x, y) ≤ d (x, z) + d(z, y) for any x, y, z ϵ X

So d is a metric on X.

Sod is a b-metric on X.

4. Hausdorff metric

In this part, we study how to define Hausdorff metric together with its equivalent metric.

4.1 Definition Let (X, d) be a metric space and CB(X) be the class of all non-empty closed and bounded subsets of X.For any A, B ϵ CB(X), put

$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$

where $d(X, B) = \inf_{y\in B} d(x, y)$, then (CB(X), H) is a metric space and H (A, B) is called a b-Hausdorff metric on CB(X). In the following, unless stated in particular H (A, B) will always denote a b-Hausdorff metric.

Remark Suppose that (X, d) is a metric space.Then H (A, B) = 0 if A=B.

4.2 Definition. Let X and Y be two non-empty subsets of a metric space (M, d), we define their Hausdorff distance $d_H(X, Y)$ by

$d_H(X, Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x, y), \sup_{y\in Y}\inf_{x\in X} d(x, y)\}$

Equivalently,

$d_H(X, Y) = \inf\{\varepsilon > 0, X \subset Y_\varepsilon \text{ and } Y \subset X_\varepsilon\}$ where $X_\varepsilon = \bigcup_{x\in X}\{z\in M / d(z,x) \le \varepsilon\}$

i.e $d_H(X, Y) = \inf\{\varepsilon > 0, X \subset Y + \varepsilon B(y, 1), Y \subset X + \varepsilon B(x, 1)\}$

$= \inf\{\varepsilon > 0, X \subset Y_\varepsilon, Y \subset X_\varepsilon\}$

$X_\varepsilon = \varepsilon$ – fattening of X = generalized ball of radius ε around X.

Let d_1 be the first metric and d_2 be the second one respectively.

Thinking of $\inf_{y\in Y} d(x, y)$ isthe distance from X to Y,

$\sup_{x\in X}\inf_{y\in Y} d(x, y)$ represents the farthest point of X from Y.

Thus $d_1(X, Y)$ gives the greatest distance a point can have from the other set. $d_2(X, Y)$ represents the smallest amount we can increase both sets to include the other. These two should clearly be equal.

Firstly, we will prove $d_1 \le d_2$ ____(1)

Take $\varepsilon \ge 0$ so that $X \subset Y_\varepsilon$, $Y \subset X_\varepsilon$.

We show $d_1(X, Y) \le \varepsilon$, and then it will follow

$d_1(X, Y) \le d_2(X, Y)$.

By symmetry, $\sup_{x\in X}\inf_{y\in Y} d(x, y) \le \varepsilon$.

So, it suffices to show $\inf_{y\in Y} d(x, y) \le \varepsilon$ for each $x \in X$.

But for any $x \in X$, $x \in Y_\varepsilon$, so is some $y \in Y$ for which $d(x, y) \le \varepsilon$ as desired.

Now, we show $d_2 \le d_1$.

Let $\varepsilon = d_1(X, Y)$.

It suffices to show $X \subset Y_{\varepsilon+\alpha}$, $Y \subset X_{\varepsilon+\alpha}$ for every $\alpha \ge 0$.

By symmetry, we just prove the first inclusion.

Take $x_0 \in X$.

Since $\sup_{x\in X}\inf_{y\in Y} d(x, y) \leq d_1(X, Y) = \varepsilon_2$ we see

$\inf_{y\in Y} d(x_0, y) < \varepsilon + \alpha$

Therefore, $x_0 \in Y_{\varepsilon+\alpha}$ as desired.

4.3 Theorem. If (X, d) is a b-metric space, then (CB(X), H) is a b-metric space.

Proof. $H(X, Y) = \inf\{\varepsilon > 0 \mid X \subset \cup_{y\in Y} B(y, \varepsilon), Y \subset \cup_{x\in X} B(x, \varepsilon)\}$.

$H(Y, X) = \inf\{\varepsilon > 0 \mid Y \subset \cup_{x\in X} B(x, \varepsilon), X \subset \cup_{y\in Y} B(y, \varepsilon)\}$.

$\therefore H(X, Y) = H(Y, X)$.

$H(X, Y) \geq 0$.

$H(X, Y) = 0 \leftrightarrow \inf\{\varepsilon > 0 \mid X \subset \cup_{y\in Y} B(y, \varepsilon), Y \subset \cup_{x\in X} B(x, \varepsilon)\} = 0$.

Let $n \in Z^+$, then $0 + \frac{1}{n} > 0$.

$\exists\, \varepsilon > 0 : \varepsilon < \frac{1}{n}$ by definition of inf and $\varepsilon \in S$.

Where $S = \{\varepsilon > 0 \mid X \subset \cup_{y\in Y} B(y, \varepsilon), Y \subset \cup_{x\in X} B(x, \varepsilon)\}$

Since $0 < \varepsilon < \frac{1}{n}$, $B(x, \varepsilon) \subset B(x, \frac{1}{n})$

$Y \subset \cup_{x\in X} B(x, \frac{1}{n})$ for all $n \in Z^+$ and

$X \subset \cup_{y\in Y} B(y, \frac{1}{n})$ for all $n \in Z^+$.

So, it is true for n: $n \to \infty$

$\cup_{x\in X} B(x, \frac{1}{n}) = X$ and

$\cup_{y\in Y} B(y, \frac{1}{n}) = Y$.

Clearly, $\cup_{x\in X} B(x, \frac{1}{n}) \subset X$ for $n \to \infty$.

Let z ϵ X.

$\exists\, x_0 \epsilon X:\ z\, \epsilon B\, (x_0, \frac{1}{n})$ for n→ ∞.

Since z ϵ B (z, $\frac{1}{n}$), choose x_0 =z.

Then $x_0 \epsilon X$ *and* z ϵ B $\left(x_0, \frac{1}{n}\right) \subset U_{x\epsilon X}$ B (x, $\frac{1}{n}$).

So X $\subset U_{x\epsilon X}$ B (x, $\frac{1}{n}$) for n → ∞.

So X ⊂Y and Y ⊂X.

∴ X = Y.

H (X, Y) = inf {ε > 0| X$\subset U_{y\epsilon Y}$ B (y, ε), Y$\subset U_{x\epsilon X}$ B (x, ε)} = γ.

H (X, Z) = inf {ε > 0| X$\subset U_{z\epsilon Z}$ B (z, ε), Z⊂ $U_{x\epsilon X}$ B (x, ε)} = α.

H (Z, Y) = inf {ε > 0| Z$\subset U_{y\epsilon Y}$ B (y, ε), Y$\subset U_{z\epsilon Z}$ B (z, ε)} = β.

$\forall\, \delta > 0, \gamma + \delta > \gamma, \exists\, \varepsilon > 0: \varepsilon < \gamma + \delta, X \subset U_{y\epsilon Y}$ B (y, ε), Y$\subset U_{x\epsilon X}$ B (x, ε).

X$\subset U_{y\epsilon Y}$ B (y, γ + δ), Y$\subset U_{x\epsilon X}$ B (x, γ + δ).

$\exists\, y_0 \epsilon X: x \epsilon B(y_0, \gamma + \delta)$

d (x, y_0) < γ + δ.

$\exists\, x_0 \epsilon X: y \epsilon B(x_0, \gamma + \delta)$

d (y, x_0) < γ + δ

$\exists\, x_1 \epsilon$ X, $z_1 \epsilon$ Z : d (x, z_1) < α + δ and d (z, x_1) < α + δ.

$\exists\, y_2 \epsilon$ Y, $z_2 \epsilon$Z : d (y_2, z) < β + δ and d (z_2, y) < β + δ.

d (x_0, y_2) ≤ d (x_0, z_1) + d (z_1, y_2).

< α + δ + β + δ.

= α + β + 2δ.

H (X, Y) = max {$\sup_{x\epsilon X}$ d(x,Y), $\sup_{y\epsilon Y}$ d(y, X)}.

d (x, Y) = $\inf_{y\epsilon Y}$ d (x, y).

d (x, Y) ≤ d (x, y).

$\sup_{x\epsilon X}$ d(x, Y) ≤ d (x, y).

Similarly, $\sup_{y\epsilon Y}$ d(y, X) ≤ d (x, y).

$\sup_{x\epsilon X}$ d(x, Y) ≤ d (x_0, y_2), $\sup_{y\epsilon Y}$ d(y, X) ≤ d (x_0, y_2).

max { $\sup_{x\epsilon X}$ d(x, Y), $\sup_{y\epsilon Y}$ d(y, X)} ≤ d (x_0, y_2) ≤ α+ β + 2δ for all $\delta > 0$.

H (X, Y) ≤ α+ β + $\frac{1}{n}$ for δ = $\frac{1}{n}$ for all $n \epsilon Z^+$.

H (X, Y) ≤ α+ β.

H (X, Y) ≤ H (X, Z) + H (Z, Y).

Hence, (CB(X), H) is a b-metric space.

Conclusion

We have surveyed on metrics such as pseudo-metric, b-metric and Hausdroff metric. Their relationships have also been investigated. Related proofs have been shown in this research paper.

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