Jordan Blocks and Jordan Form of a Complex Matrix

MyintMyint Maw*

Abstract

In this paper we study one of the most important results in linear algebra – the Jordan form of a complex matrix A under similarity transformations $A \to S^{-1}AS$. First we start the preliminary definitions and the Jordan block. Then we discuss the construction of similarity transformation, it can be change into a direct sum of Jordan blocks. Illustrations are also given.

Keyword: Eigenvalue, Eigenvector, Jordan chain, Jordan block, invariant subspace

1.Construction of Jordan Form

Let $A: C^n \to C^n$ be a linear operator.

1.1Definition: A number $\lambda \in C$ is called an **eigenvalue** of A if there exists $x \in C^n$ such that $x \neq 0$ and $Ax = \lambda x$. The vector x is called an **eigenvector** of A corresponding to λ.

1.2 Definition. A subspace $M \subset C^n$ is called an **invariant subspace** for the operator A or A-invariant if $Ax \in M$ for every vector $x \in M$.

1.3Definition. Let λ be an eigenvalue of a linear operator A. A chain of vectors x_0 , x_1 , …, x_k is called **Jordan chain** of A corresponding to λ if $x_0 \neq 0$ and the following relation hold: $Ax_0 = \lambda x_0$, $Ax_1 - \lambda x_1 = x_0$, …, $Ax_k - \lambda x_k = x_{k-1}$.

1.4Definition. An A-invariant subspace M is called a **Jordan subspace** corresponding to the eigenvalue λ of A if M is spanned by the vectors of some Jordan chain of A corresponding to λ.

1.5Definition. Let A be an $n \times n$ complex matrix and λ_0 be an eigenvalue of A.

The **Jordan block** of size $k \times k (k \leq n)$ with eigenvalue λ_0 is the matrix

$$J_k(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & 1 \\ 0 & 0 & 0 & \cdots & \lambda_0 \end{bmatrix}$$ Clearly $\det(\lambda I - J_k(\lambda_0)) = (\lambda - \lambda_0)^k$.

So λ_0 is the only eigenvalue of $J_k(\lambda_0)$. $\lambda_0 I - J_k(\lambda_0) = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & -1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$

So the only eigenvector of $J_k(\lambda_0)$ (up to multiplication by a non-zero complex number) is $e_1 = (1, 0, \ldots, 0)$.

1.6Theorem. Let $A = J_k(\lambda_0)$. Then every nonzero A-invariant subspace is of the form *Span* $\{e_1, \ldots, e_j\}$, where e_i is the vector (0, …, 0, 1, 0, …,) with 1 in the i^{th} place.

* Lecturer, Department of Mathematics, Dagon University

Proof. Let Mbe a non-zero A-invariant subspace and let $x=\sum_{i=1}^{k}\alpha_i e_i, \alpha_i \in C$ be a vector of M for which the index j = max $\{m|\ 1 \le m \le k,\ \alpha_m \ne 0\}$ is maximal.

Thus $M \subset Span\ \{e_1, \ldots, e_j\}$.

On the other hand, consider $x=\sum_{i=1}^{j}\alpha_i e_i \in M, \alpha_j \ne 0.$

Since M is A-invariant

$$x_1 = Ax-\lambda_0 x=(A-\lambda_0 I)x=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & \vdots\\ \vdots & \vdots & \vdots & & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_j\\ 0\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}\alpha_2\\ \alpha_3\\ \vdots\\ \alpha_j\\ 0\\ \vdots\\ 0\end{bmatrix}=\sum_{i=2}^{j}\alpha_i e_{i-1}\in M$$

$$x_2 = Ax_1-\lambda_0 x_1=(A-\lambda_0 I)x_1=\sum_{i=3}^{j}\alpha_i e_{i-2}\in M$$

$$\vdots$$

$$x_{j-1}=Ax_{j-2}-\lambda_0 x_{j-2}=(A-\lambda_0 I)x_{j-2}=\alpha_j e_1\in M$$

Hence

$$e_1 = \frac{1}{\alpha_j}x_{j-1}\in M$$

$$e_2 = \frac{1}{\alpha_j}(x_{j-2}-\alpha_{j-1}e_1)\in M$$

$$\vdots$$

$$e_j = \frac{1}{\alpha_j}\left(x-\sum_{i=1}^{j-1}\alpha_i e_i\right)\in M.$$

so$Span\ \{e_1, \ldots, e_j\} \subset M$.Thus$Span\ \{e_1, \ldots, e_j\} = M$.

1.7Theorem. Let A be an$n \times n$ complex matrix. Then there exists an invertible matrix S such that $S^{-1}AS$ is a direct sum of Jordan blocks (Jordan form)

(1) $$S^{-1}AS = J_{k_1}(\lambda_1)\oplus\ldots\oplus J_{k_p}(\lambda_p)=\begin{bmatrix}J_{k_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{k_p}(\lambda_p)\end{bmatrix}$$

$\lambda_1, \ldots, \lambda_p$ are eigenvalues of A.

Proof. Let $J=J_{k_1}(\lambda_1)\oplus\ldots\oplus J_{k_p}(\lambda_p),\ k_1+k_2+\ldots+k_p=n$

Clearly we see that J has eigenvalues $\lambda_1, \ldots, \lambda_p$.

(1) becomes $S^{-1}AS = J$.

Then we rewrite in the form $AS = SJ$

$$A\begin{bmatrix} x_1 x_2 \dots x_n \end{bmatrix} = \begin{bmatrix} x_1 x_2 \dots x_n \end{bmatrix}\begin{bmatrix} J_{k_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{k_p}(\lambda_p) \end{bmatrix}$$

where x_1 , x_2 , … , x_n are columns of S. Then we have

$Ax_1 = \lambda_1 x_1,\quad Ax_2 = \lambda_1 x_2 + x_1,\quad \dots,\quad Ax_{k_1} = \lambda_1 x_{k_1} + x_{k_1 - 1}$

$Ax_{k_1+1} = \lambda_2 x_{k_1+1},\quad Ax_{k_1+2} = \lambda_1 x_{k_1+2} + x_{k_1+1},\quad \dots,\quad Ax_{k_1+k_2} = \lambda_1 x_{k_1+k_2} - x_{k_1+k_2-1}$

$\vdots$

$Ax_{k_1+\dots+k_{p-1}+1} = \lambda_p x_{k_1+\dots+k_{p-1}+1},\quad \dots\quad Ax_n = \lambda_p x_n + x_{n-1}$

Thus the search for the Jordan form of A becomes a search for Jordan chains of A. So we shall construct invertible matrix S containing Jordan chains column vectors. For every i

(2) either $Ax_i = \lambda_i x_i$ or $Ax_i = \lambda_i x_i + x_{i-1}$

We use mathematical induction.

Every 1 × 1 matrix is in its Jordan form.

We assume that the construction of Jordan chains is satisfied for all matrices of order less than n.

Let $A: C^n \to C^n$.

Step 1. If A is singular, then its column space of A, $C(A)$ (or) range of A is smaller subspace of C^n. So $\dim C(A) = r < n$. Looking only within this smaller space, the induction hypothesis guarantees that a Jordan form is possible. There exists r independent vectors w_i in the column space such that either $Aw_i = \lambda_i w_i$ or $Aw_i = \lambda_i w_i + w_{i-1}$.

Step 2. Let $\dim (Ker\,(A) \cap C(A)) = p$

If $x \in Ker\,(A)$, then $Ax = 0 = 0x$ so x is an eigenvector corresponding to $\lambda = 0$.

Thus there exists p Jordan chains with $\lambda = 0$ in step 1.

Next we consider the vectors w_i they are at the end of these p Jordan chains.

Since $w_i \in C(A)$, each w_i is combination of the columns of A.

So $w_i = Ay_i$ for some y_i. Thus $Ay_i = 0y_i + w_i$.

Now each y_i is at the end of these p Jordan chains respectively.

Step 3. By fundamental theorem of linear transformation

$$\dim C^n = \dim Ker\,(A) + \dim C(A)$$

$$n = \dim Ker\,(A) + r$$

So $\dim Ker\,(A) = n - r$

$\dim (Ker\,(A) - C(A)) = n - r - p$

Let z_1 , … z_{n-r-p} be a basis vectors of Ker $(A) - C(A)$.

These vectors are eigenvectors corresponding to $\lambda = 0$.

Each one exists alone in its own Jordan chain.

The r vectors w_i, the p vectors $y_i$and the $n - r - p$ vectors z_i form Jordan chains for matrix A. If these vectors are linearly independent, then we put these vectors into S as columns.

Now we shall show that the vectors w_i, y_i and z_i are linearly independent.

(3) Let $\sum c_i w_i + \sum d_i y_i + \sum g_i z_i = 0$ for some scalars c_i ,d_i and g_i.

Multiplying by A and using equation (2) for the w_i and $Az_i = 0$,

(4) $$\sum c_i \begin{bmatrix} \lambda_i w_i \\ \text{or} \\ \lambda_i w_i + w_{i-1} \end{bmatrix} + \sum d_i A y_i = 0$$

$Ay_i = w_i$ at the ends of Jordan chains corresponding to $\lambda_i = 0$.

By induction hypothesis w_i are independent. So each $d_i = 0$.

Thus (3) becomes $\sum c_i w_i = -\sum g_i z_i$ and left-hand side is in column space.

Since z_i is not in column space, $g_i = 0$. Thus (3) becomes $\sum c_i w_i = 0$.

Since w_i are independent vectors, $c_i = 0$. Thus S is invertible.

If A is not singular, then we can use any one eigenvalue of A, say c. We have proved that $A - cI$is not invertible.

Thus $A - cI = A'$ is singular.

Jordan form is $S^{-1}A'S = J'$.

$$S^{-1}AS = S^{-1}A'S + S^{-1}cS = J' + cI = J$$

Jordan form of Auses the same Jordan chains and same S.

1.8 Example. Let us consider the following matrix

$$A = \begin{bmatrix} 8 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

Clearly 8, 8, 0, 0, 0 are eigenvalues of A.

Step 1. We can reduce matrix A to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Thus *Span* $\{e_1, e_2, e_5\} = C(A)$ and $\dim C(A) = r = 3$.

If we ignore the third and fourth rows and columns of A, then left eigenvalues are 8, 8, 0. We choose 3 independent column vectors in $C(A)$.

$$w_1 = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in C(A)$$

Complete Jordan chain for $\lambda = 8$ is w_1, w_2.

So $\quad Aw_1 = 8w_1, Aw_2 = 8w_2 + w_1$

Beginning vector(s) of Jordan chian for $\lambda = 0$ is w_3.

So $\quad Aw_3 = 0w_3$

Step 2. Since $Ae_2 = 0$, $Ae_3 = 0$, $e_2, e_3 \in Ker\ (A)$.

Thus $Ker\ (A) \cap C(A) = Span\ \{e_2\}$.

So there is one Jordan chain corresponding to $\lambda = 0$ in step 1.

Since $w_3 \in Ker\ (A) \cap C(A)$,

$w_3 = Ay$ for some y

$A(e_4 - e_1) = w_3$. So $y = e_4 - e_1$.

Step 3. $n - r - p = 5 - 3 - 1 = 1 = \dim\ (Ker\ (A) - C(A))$ and $z = e_3 \in Ker\ (A) - C(A)$

$Az = Ae_3 = 0 = 0z$

Thus z produces 1×1 block in J.

For all five vectors, the Jordan chains are

$Aw_1 = 8w_1, Aw_2 = 8w_2 + w_1, Aw_3 = 0w_3, Ay = 0y + w_3, Az = 0z$

Putting the five vectors into the columns of S

$$S = \begin{bmatrix} 8 & 0 & 0 & -1 & 0 \\ 0 & 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$ Then we can check that $AS = SJ$

where
$$J = \begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 8 & 1 \\ 0 & 8 \end{bmatrix} & & \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \\ & & [0] \end{bmatrix} = \begin{bmatrix} J_2(8) & 0 & 0 \\ 0 & J_2(0) & 0 \\ 0 & 0 & J_1(0) \end{bmatrix}$$

2. One Application of Jordan Block

Let us consider simple problem in nature. We have three V gallon tanks as shown in Figure that are initially full of polluted water in which the i[th] tank contains $c_i$lb of a pollutant. In an attempt to flush the pollutant out, all spigots are opened at once allowing fresh water at the rate of $r$gal/sec to flow into the top of tank #3, while r gal/sec flow from its bottom into the top of tank #2, and so on.

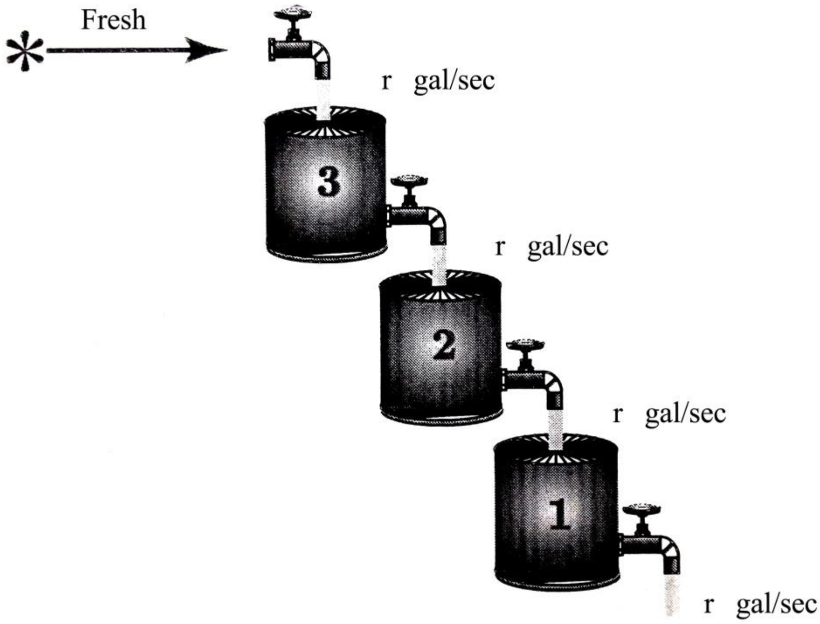

Fig. (1)

Now we are interested in how many pounds ofthe pollutant are in each tank at any finite time $t > 0$ when instantaneous and continuous mixing occurs?

If $u_i(t)$ denotes the number of pounds of pollutant in tank i at time $t > 0$, then the pollutant in tank i at time t is $u_i(t)/V$lbs /gal, so the model

$u'_i(t)$ = (lbs/sec) coming in -(lbs/sec) going out.

$$\begin{pmatrix} u'_1(t) \\ u'_2(t) \\ u'_3(t) \end{pmatrix} = \frac{r}{V}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}$$ or $u' = Au$ with $u(0) = c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.$A$ is simply a scalar multiple of a single Jordan block $J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, so e^{At} is determined by replacing t by rt/V and λ by -1. Then

$$e^{At} = e^{(rt/V)J} = e^{-rt/V}\begin{pmatrix} 1 & \frac{rt}{V} & (\frac{rt}{V})^2/2 \\ 0 & 1 & \frac{rt}{V} \\ 0 & 0 & 1 \end{pmatrix}.$$

So $$u(t) = e^{At}c = e^{-rt/V}\begin{pmatrix} c_1 + c_2\left(\frac{rt}{V}\right) + c_3(\frac{rt}{V})^2/2 \\ c_2 + c_3(\frac{rt}{V}) \\ c_3 \end{pmatrix}.$$

(See in [2] Page 599, 7.9 Functions of Nondiagonalizable Matrices)

References

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