Statistical Law for Arrival Rate of a Roadway

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Abstract

First we used the Chi-square hypothesis testing method to test the number of arrived vehicles at Anawrahta roadway (TheinPhyu to Bo AungKyaw) for the input process obeys the Poisson distribution. Then, we observed the traffic intensity of the Anawrahta-Bo AungKyaw stop point. We found that there is enough to set the red light-green light durations to none congestion.

1. Distribution for Arrival Rate

1.1 Introduction

Queueing theory is the mathematical theory and method of queueing system. In daily life, people will encounter all sorts of queueing problems such as standing at bus stops, traffic congestion on the road, going to hospital, and going to the ticket office to buy the tickets and so on. In these problems, buses and passengers, vehicles and human, doctors and patients, conductors and the buyers form a queueing system.

The queue can be tangible queue and may also be intangible queue. For example, several passengers make telephone call to order car tickets at the same time, if a passenger is on the phone, can only wait for the other passengers, this form of queue is invisible.

A queueing system can be described as three parts: customers arriving for service, waiting for service and leaving the system after being served.

All queueing systems can be completely described and classified according to the following basic characteristics.

(1) Arrival pattern of customers.
(2) Service pattern of servers.
(3) Queue discipline.
(4) System capacity.

Queueing theory can be divided into single channel queueing system and multi-channel queueing system.

1.2 Performance Indicators for Single Channel Queueing System

Combined with the Little's formula, the quantity indexes of single channel queueing system can be obtained, as follows:

(1) The probability of no vehicle in the system:

$$P_0 = 1-\rho.$$

(2) The probability of *n* vehicles in the system:

$$P_n = \rho^n (1-\rho).$$

(3) The average number of vehicles in the system:

$$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}.$$

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(4) The average queueing length of vehicles in the system:

$$L_q = \rho L_s = \frac{\lambda^2}{\mu(\mu-\lambda)}.$$

(5) The average staying time of vehicles in the system:

$$W_s = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{1}{\mu-\lambda}.$$

(6) The average waiting time of vehicles in the system:

$$W_q = W_s - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

1.3 Maximum Likelihood Estimator

First, it needs to estimate the parameter λin Poisson distribution by the maximum likelihood method.

Assume the whole $X \sim Pois(\lambda)$:

(1) $$P(X = x_i) = \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}, x_i = 0,1,2, \ \dots \ .$$

Then the *likelihood function* of parameterλ:

(2) $$L(\lambda) = \prod_{i=1}^{n} P(X = x_i)$$

$$= \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$$

$$= \frac{e^{-n\lambda}}{x_1!\, x_2! \dots x_n!} \lambda^{x_1+x_2+\dots+x_n}$$

Taking logarithm on both sides, we have

$$\ln(L(\lambda)) = \ln\left(\frac{e^{-n\lambda}}{x_1! x_2! \dots x_n!} \lambda^{\sum x_i} \right)$$

$$= \ln e^{-n\lambda} + \ln \lambda^{\sum x_i} - \ln(x_1! x_2! \dots x_n!)$$

$$= -n\lambda + \sum x_i \ln \lambda - \ln(x_1! x_2! \dots x_n!)$$

Differentiating with respect to λ, the likelihood equation is obtained:

(3) $$\frac{d \ln(L(\lambda))}{d\lambda} = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i \ .$$

For stationary, we let $\frac{d \ln(L(\lambda))}{d\lambda} = 0$ thus we have

(4) $$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}.$$

Also,

(5) $$\frac{d^2 \ln(L(\lambda))}{d\lambda^2}\Big|_{\lambda=\bar{x}} = \frac{-n\ \bar{x}}{\lambda^2}\Big|_{\lambda=\bar{x}} = \frac{-n}{\bar{x}} < 0.$$

So the *maximum likelihood estimator* of parameter $\lambda$is $\hat{\lambda} = \bar{x}$.

1.4 Chi-Square Hypothesis Testing

In this section, we will validate the number of arrival vehicles to an intersection obeys the Poisson distribution by using χ^2 hypothesis testing method. Thus, we take the traffic intersections of Yangon District for example, especially, the Anawrahta road (TheinPyu - Bo AungKyaw section). We note the number of vehicles in East-West direction when the traffic lights change each cycle.

Table (1) Number of Vehicles Arriving at 1 Traffic Light Cycle

On Anawrahta Road (TheinPyu - Bo AungKyawsection)

5:01 am	6	5	6	7	6
to	8	7	8	8	9
6:00 am	7	8	9	8	11
6:01 am	11	12	12	8	9
to	11	11	13	12	12
7:00 am	12	11	10	14	17
7:01 am	12	14	14	15	16
to	14	15	16	19	20
8:00 am	19	18	17	18	21

From the above raw data we have the sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{536}{45} = 11.91.$$

Thus the average arrival rate $\lambda = \bar{x} = 11.91$.

The above data can be divided into 6 groups, 5 -7, 8 - 10, 11 - 13, 14 - 16, 17 - 19, 20 - 22. Here f_i is the frequency that belongs to the i^{th} group of vehicles. See the fourth column of table (2). The probability of i^{th} group is given by

$$P_i = \sum_{k=a_{i-1}}^{a_i} \frac{\lambda^k}{k!} e^{-\lambda}$$

where a_{i-1} is the lower bound and a_i is the upper bound of the i^{th} group. $f_i^* = P_i \sum_{i=1}^{6} f_i = 45P_i$ is the theoretical frequency, λ is the average number of arrival vehicles.

Table (2) Frequency Table

Lower class boundary a_{i-1}	Class Mid point x_i	Upper class boundary a_i	Observed Frequency f_i	Probability Pi	Expected Frequency f_i^*	$(f_i - f_i^*)^2/f_i^*$
5	6	7	7	0.085	4	2.591
8	9	10	10	0.263	13	0.287
11	12	13	12	0.334	15	0.615
14	15	16	8	0.213	9	0.255
17	18	19	6	0.077	3	1.889
20	21	22	2	0.017	1	1.971
		Total	45	0.999	45	**7.608**

From the above data and formulas, we can calculate the χ^2 number

$$\chi^2 = \sum_{i=1}^{6} \frac{(f_i - f_i^*)^2}{f_i^*} \approx 7.608.$$

Because of estimating a parameter λ when calculating the probability, m =1. The degree of freedom is $\upsilon = N-1-m = 6-1-1 = 4$, α=0.05 is selected, referring the Chi-square distribution table,

$$\chi^2_{0.95}(4) = 9.488.$$

Thus we have $\chi^2 = 7.608 < 9.488 = \chi^2_{0.95}(4)$.

Here, the Chi-square value is less than $\chi^2_{0.95}(4)$.So the hypothesis "*the number of the arrived vehicles per unit time obeys the Poisson distribution*" is valid.

For accuracy, the Chi-square value χ^2 is greater than $\chi^2_{0.05}(4) = 0.71$, so the fit is not too good.

This survey has only done for the arrival rate at Anawrahta one-way road. And for the other 2 directions, we can verify through the same method, but their parameters are different.

2. Probability for Non-congested Flow

2.1 Intensity of the Section

In this section we observed some performance indicators for adjusting the signal light cycle to get a free flow (non-congested flow) by using probability theory. We used the above sample data to calculate the following table and it shows the probable time durations of light cycle to get low intensity.

The intensity is defined by $\rho = \lambda / \mu$ where λ is the arrival rate and μ is the service rate of roadway. Here, ρ must be less than 1, if $\rho \geq 1$, we cannot use the queueing theory.

The probability of no vehicle in the roadway section is defined by

$$P_0 = 1-\rho;\ \ 0<\rho< 1.$$

Thus, $P_0 \to 1$ as $\rho \to 0$.

We now adjust the red light - green light cycle for uncongested flow at the Bo AungKyaw intersection. If the data from table (1) are used, we have the following table.

Table (3) Red light-Green light table for none congested flow

Surveying Time	Intensity (λ/μ)	Probability	Red (sec)	Green (sec)
5:00 am to 6:00 am	0.0114	0.989	**15**	**15**
7:00 am to 8:00 am	0.0153	0.985	**20**	**20**
9:00 am to 10:00 am	0.0262	0.974	**35**	**30**

2.2 Conclusion

We notice that,in statistical law,corresponding data of an event is needed to correct. By Chi-square testing,we can decide the probability distribution for arrival rate. We can adjust the red light - green light in corresponding time interval for uncongested flow by using probability theory.

Our goal is to solve the traffic congested problems by usingqueueing theory.

References

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