Bounded Linear 2-functionals in Linear 2-normed Space

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Abstract

In this paper, we introduce the linear 2-normed spaces with examples and then we present some of their properties. Next we describe the bounded linear 2-functionals in linear 2-nomred spaces. Finally, we claim that the space of these functional is a Banach space.

1. Introduction

We already know that the basic properties of bounded linear functionals in normed linear spaces and the set of these functionals is a BanachSpace.In 1964, S.Gahler introduced the concepts of linear 2-normed spaces and their topological structures. In 2004, Z,Lewandowska established some properties of 2- linear operators on 2- normed sets. In this paper, we described some properties of bounded linear 2-functionals in linear 2-normed spaces and also proved that the set of bounded linear 2-functionals is a Banach Space.

2.Linear 2-normed space

2.1Definition

Let X be a real linear space of dimension greater than 1 and $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions:

($2N_1$) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

($2N_2$) $\|x, y\| = \|y, x\|$,

($2N_3$) $\|\alpha x, y\| = |\alpha|\|x, y\|$,

($2N_4$) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$and $\alpha \in R$.

$\|.,.\|$is called a **2-norm** on X and $(X, \|.,.\|)$ is called a **linear 2-normed space.**

Then we have the following basic properties of 2- norm.

2.2 Theorem

Let $(X, \|.,.\|)$ be a linear 2-normed space.For all $x, y, z \in X$ and for every realα , then

(i) $\|x, y\| \geq 0$

(ii) $\|x, y + \alpha x\| = \|x, y\|$

(iii) $\|x - z, x - y\| = \|x - z, y - z\|$.

(iv) $\|x + z, y + z\| \leq \|x, y\| + \|y, z\| + \|z, x\|$, for all $x, y, z \in X$.

2.3 Example

Let $X = R \times R \times R$ with 2-norm defined as follows:

Let$\|x, y\| = |b_1c_2 - b_2c_1| + |a_1c_2 - a_2c_1| + |a_1b_2 - a_2b_1|$ for $x = (a_1, b_1, c_1)$ and

$y = (a_2, b_2, c_2)$in X.Let vector addition and scalar multiplication be defined componentwise. Then the 2-norm properties are satisfied.

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2.4 Example

Let $(X.\|\,.\,\|)$ be a real normed vector space. Then $(X,\|.,.\|)$ is not a 2-normed space with the 2-norm defined by the formula $\|x,y\| = \|x\|.\|y\|$ for each $x, y \in X$.

For let $x \neq 0, y = kx$ and $k \neq 0$.Then $\|x,y\| = \|x,kx\| = \|x\|\|kx\| = |k|\|x\|^2 > 0$.

2.5 Example

Let E^3 denote Euclidean 3-dimensional linear space. Let $x = ai + bj + ck$ and

$y = di + ej + fk$. Define

$$\|x,y\| = |x \times y| = abs \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix}$$

$$= |(bf - ce)i + (cd - af)j + (ae - db)k|$$

$$= [(bf - ae)^2 + (cd - af)^2 + (ae - db)^2]^{1/2}$$

From the vector analysis, it is clear that $\|.,.\|$ is a 2-norm.

2.6 Theorem

Let $(X,\|.,.\|)$ be a 2-normed space. For $a = \alpha_1 e_1 + \alpha_2 e_2$ and $b = \beta_1 e_1 + \beta_2 e_2$. with basis $\{e_1, e_2\}$, we have $\|a,b\| = |\alpha_1\beta_2 - \beta_1\alpha_2|\|e_1,e_2\|$.

2.7Theorem

Consider the norm $\|\,.\,\|$ defined on a linear 2-normed space $(X,\|.,.\|)$ by the function $\|a\| = \|a,y\| + \|a,z\|$,for any fixed $y, z \in X$ and $\|y,z\| \neq 0$.

Then the function $\|\,.\,\|$ is a norm on X.

Proof.

(i) $\|a\| \geq 0$.

If $\|a\| = 0$, then $\|a,y\| = 0$ and $\|a,z\| = 0$.

Under the assumption that $a \neq 0$, this implies that $a = \alpha y$ and $a = \beta z$ with $\alpha \neq 0$and $\beta \neq 0$, which contradicts the linear independence of y and z.

Thus $\|a\| = 0$ implies that $a = 0$.

(ii) $\|a+b\| = \|a+b,y\| + \|a+b,z\| \leq \|a,y\| + \|b,y\| + \|a,z\| + \|b,z\| = \|a\| + \|b\|$.

(iii) $\|\alpha a\| = \|\alpha a,y\| + \|\alpha a,z\| = |\alpha|(\|a,y\| + \|a,z\|) = |\alpha|\|a\|$.

3. Bounded Linear 2-Functionals

3.1 Definition

A **2-functional** is a real valued mapping with domain $A \times C$, where A and C are linear subspace of a linear 2-normed space.

3.2 Definition

Let F be a 2-functional with domain $\times C$,where A and C are linear subspaces of a linear 2-normed space.Then F is called a **linear 2-functional** (or **bilinear 2-functional**) if

(1) $F(a+c,b+d)=F(a,b)+F(a,d)+F(c,b)+F(c,d)$.

(2) $F\ (\alpha a,\beta b)\ =\ \alpha\beta\ F(a,b)for\alpha,\beta$ in the underlying field.

3.3 Definition

Let F be a 2-functional with domain $D(F).F$ is said to be **bounded** if there is a real constant $K\geq 0$ such that $|F(a,b)|\leq K\|a,b\|$ for all $(a,b)\in D(F)$.

If F is bounded, define the norm of $F,\|F\|$ by $\|F\|=glb\ \{|K\!: F(a,b)|\leq K\ \|a,b\|$ for all $(a,b)\in D(F)\}$.If F is not bounded, define $\|F\|=+\infty$.

3.4 Example

Let $(X,\|.,.\|)$ be a linear 2-normed space with basis $\{e_1,e_2\}$.

Define $F(a,b)=\alpha_1\beta_2-\alpha_2\beta_1$ where $a=\alpha_1e_1+\alpha_2e_2$ and $b=\beta_1e_1+\beta_2e_2$.

Let $c=\mu_1e_1+\mu_2e_2$ and $d=\delta_1e_1+\delta_2e_2$. Then we have

$$
\begin{aligned}
F(a+c,b+d) &= (\alpha_1+\mu_1)\,(\beta_2+\delta_2)-(\alpha_2+\mu_2)\,(\beta_1+\delta_1)\\
&= \alpha_1\beta_2+\alpha_1\delta_2+\mu_1\beta_2+\mu_1\delta_2-\alpha_2\beta_1-\alpha_2\delta_1-\mu_2\beta_1-\mu_2\delta_1\\
&= (\alpha_1\beta_2-\alpha_2\beta_1)+(\alpha_1\delta_2-\alpha_2\delta_2)-(\mu_1\beta_2-\mu_2\beta_1)+(\mu_1\delta_2-\mu_2\delta_1)\\
&= F(a,b)+F(a,d)+F(c,b)+F(c,d).
\end{aligned}
$$

$$F(\alpha a,\beta b)=\alpha\alpha_1\beta\beta_2-\alpha\alpha_2\beta\beta_1=\alpha\beta(\alpha_1\beta_2-\alpha_2\beta_1)=\alpha\beta\ F(a,b)$$

and$|F(a,b)|=|\alpha_1\beta_2-\alpha_2\beta_1|=\frac{1}{\|e_1,e_2\|}\|a,b\|$.

Therefore, F is a bounded linear 2-functional.

3.5 Example

Let $(E^3,\|.,.\|)$ be the 2-normed space define in example (2.5). Define $F(x,y)=(x|y)$ where $(.|.)$ is the dot product of vector analysis. Then F is an unbounded linear 2-functional. Define $G(x,y)=(|x|^2|y|^2-|(x|y)|^2)^{1/2}$,where $|a|$ denotes the length of a. Then G is a bounded 2-functional since $|x|^2|y|^2-|(x|y)|^2=|x\times y|^2$.

3.6 Lemma.

If F is a bounded linear 2-functional, and a and b are linearly dependent with $(a,b)\in D(F)$ then $F(a,b)=0$.

Proof. Since F is bounded, $|F(a,b)|\leq\|F\|\|a,b\|$ for all $(a,b)\in D(F)$.

Since a an b are linearly dependent, $\|a,b\|=0$.

Therefore we have $|F(a,b)|\leq\|F\|.0=0$.

3.7 Theorem

Let F be a bounded linear 2-functional with domain $D(F)$. Then we have

$$\|F\| = sup\{|F(x,y)|: \|x,y\| = 1, (x,y) \in D(F)\}$$
$$= sup\left\{\frac{|F(x,y)|}{\|x,y\|}: \|x,y\| \neq 0, (x,y) \in D(F)\right\}.$$

Proof. Let $A = sup\{|F(x,y)|: \|x,y\| = 1, (x,y) \in D(F)\}$

Then we have $|F(x,y)| \leq \|F\|\|x,y\|$for all $(x,y) \in D(F)$.

Hence $A \leq \|F\|$.Assume that $\|x,y\| \neq 0$. Since $\left\|\frac{x}{\|x,y\|}, y\right\| = 1, \left|F\left(\frac{x}{\|x,y\|}, y\right)\right| \leq A$.

Therefore, $|F(x,y)| \leq A\|x,y\|$ for $(x,y) \in D(F)$ with $\|x,y\| \neq 0$.

If $\|x,y\| = 0$, then x and y are linearly dependent. By lemma(3.6), $F(x,y) = 0$.

Thus,$|F(x,y)| \leq A\|x,y\|$ for all $(x,y) \in D(F)$ and hence we have $\|F\| \leq A$.

Let $C = sup\left\{\frac{|F(x,y)|}{\|x,y\|}: \|x,y\| \neq 0, (x,y) \in D(F)\right\}$

By the definition of $\|F\|$, we have $|F(x,y)|/\|x,y\| \leq \|F\|$for all $(x,y) \in D(F)$ with $\|x,y\| \neq 0$. So we have $C \leq \|F\|$.

$|F(x,y)| \leq C\|x,y\|$for all $x, y \in D(F)$, be lemma (3.6) and the definition of C.

Hence we have $\|F\| \leq C$.

3.8 Definition

A 2-functional $F$is said to be **continuous at** $(\boldsymbol{a}, \boldsymbol{b})$ if given $\varepsilon > 0$ there is a $\delta > 0$ such that $|F(a,b) - F(c,d)| < \varepsilon$ whenever $\|a - c, b\| < \delta$ and $\|c, b - d\| < \delta$ or

$\|a - c, d\| < \delta$and$\|a, b - d\| < \delta. F$ is **continuous** if it is continuous at each point of its domain.

3.9 Theorem

The 2-norm $\|.,.\|$ is a continuous 2-functional.

Proof. By the property (N_4) of the 2-norm $\|.,.\|$, we have

$$\|a,b\| = \|a - c + c, b\| \leq \|a - c, b\| + \|c, b\| = \|a - c, b\| + \|c, (b - d) + d\|$$

$\leq \|a - c, b\| + \|c, b - d\| + \|c, d\|$,it follows that$\|a,b\| - \|c,d\| \leq \|a - c, b\| + \|c, b - d\|$.

On the other hand, we have $\|c,d\| = \|c, (d - b) + b\| \leq \|c, d - b\| + \|c, b\|$

$$= \|c, d - b\| + \|(c - a) + a, b\| \leq \|c, d - b\| + \|c - a, b\| + \|a, b\|,$$

it follows that $\|c,d\| - \|a,b\| \leq \|a - c, b\| + \|c, b - d\|$.

Thus, we have $|\|a,b\| - \|c,d\|| \leq \|c, b - d\| + \|a - c, b\|$

Therefore, the 2-norm $\|.,.\|$is a continuous 2-functional.

3.10 Theorem

If a linear 2-functional F is continuous at $(0,0)$, then it is continuous at each point in its domain $D(F)$.

Proof. Note that if F is linear, then $F(0,0) = 0$.

Since F is continuous at $(0,0)$, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|F(c,d)| < \frac{1}{2}\varepsilon$ whenever $\|c,d\| < \delta$.

Let $(a,b) \in D(F)$. Then we have $|F(a,b) - F(x,y)| = |F(a,b) - F(x,b) + F(x,b) - F(x,y)|$

$$\leq |F(a,b) - F(x,b)| + |F(x,b) - F(x,y)|$$

$$= |F(a-x,b)| + |F(x,b-y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

wherever $\|a-x,b\| < \delta$ and $\|x,b-y\| < \delta$. Hence F is continuous at (a,b).

3.11 Theorem

A linear 2-functional F is continuous if and only if it is bounded.

Proof. Assume that F is continuous.

There exist a $\delta > 0$ such that $|F(a,b)| < 1$ whenever $\|a,b\| < \delta$ for $(a,b) \in D(F)$.

For $(c,d) \in D(F)$ with c and d being linearly independent, consider $((c/\|c,d\|)(\delta/2), d)$.

Then we have $\left\|\frac{c}{\|c,d\|}\frac{\delta}{2}, d\right\| = \frac{1}{\|c,d\|}\frac{\delta}{2}\|\mathrm{c},\mathrm{d}\| = \frac{\delta}{2}$.

Hence we have $F((c/\|c,d\|)(\delta/2), d) < 1$, that is, $|F(c,d)| < \left(\frac{2}{\delta}\right)\|c,d\|$.

There exists a $\delta_n > 0$ such that $|F(a,b)| < \frac{1}{n}$ whenever $\|a,b\| < \delta_n$.

If c and d are linearly dependent, $\|c,d\| = 0 < \delta_n$ and hence $|F(c,d)| = 0$.

Therefore, F is bounded.

Conversely, assume that F is bounded.

There is a $K \geq 0$ such that $|F(x,y)| \leq K\|x,y\|$ for all $(x,y) \in D(F)$.

Given $\varepsilon > 0$, let $\delta = \varepsilon/(K+1)$.

Then we have $|F(x,y)| \leq K\|(x,y)\| < K\frac{\varepsilon}{K+1}$ whenever $\|x,y\| < \delta$.

Hence F is continuous at $(0,0)$ and so F continuous by theorem (3.10).

3.12 Definition

Let $(X, \|.,.\|)$ be a 2-normed space and X^* be the set of bounded linear 2-functionals with domain $X \times X$. Let $F, G \in X^*$. Define

(1) $F = G$ if $F(a,b) = G(a,b)$,

(2) $(F+G)(a,b) = F(a,b) + G(a,b)$,

(3) $(\alpha F)(a,b) = \alpha F(a,b)$ for all $(a,b) \in X \times X$

By using definition (3.3) and (3.12) we have the followingtheorem .

3.13 Theorem

$(X^*, \|\,.\,\|)$is a Banach space.

Proof. For $F, G \in X^*, (a,b),(c,d) \in X \times X$ and $\alpha \in K$.

$$(F+G)(a+c, b+d) = F(a+c,b+d) + G(a+c,b+d)$$
$$= F(a,b) + F(a,d) + F(c,b) + F(c,d) + G(a,b) + G(a,d)$$
$$+G(c,b) + G(c,d)$$
$$= (F(a,b) + G(a,b)) + F(a,d) + G(a,d)) + (F(c,b) + G(c,b))$$
$$+(F(c,d) + G(c,d))$$
$$= (F+G)(a,b) + (F+G)(a,d) + (F+G)(c,b) + (F+G)(c,d).$$
$$(F+G)(\alpha a, \beta b) = F(\alpha a, \beta b) + G(\alpha a, \beta b) = \alpha\beta F(a,b) + \alpha\beta G(a,b)$$
$$= \alpha\beta\big(F(a,b) + G(a,b)\big) = \alpha\beta(F+G)(a,b).$$
$$|(F+G)(a,b)| = |F(a,b) + G(a,b)| \leq |F(a,b)| + |G(a,b)| \leq \|F\|\|(a,b)\| + \|G\|\|a,b\|$$
$$= (\|F\| + \|G\|)\|a,b\|$$

Therefore, $F + G \in X^*$ and $\|F+G\| \leq \|F\| + \|G\|$.

Similarly, $\alpha F \in X^*$.Hence, X^* is a linear space.

$\|\,.\,\|$defines a norm on X^*, since we have

(1) If $\|F\| = 0$, then $F = 0$. If $F = 0$, then $\|F\| = 0$

(2) $\|\alpha F\| = |\alpha|\|F\|$,

(3) $\|F+G\| \leq \|F\| + \|G\|$by the above argument.

Assume that $\{F_n\}$ is a Cauchy sequence in X^*.Then for every $\varepsilon > 0$, there is an integer N such that $\|F_m - F_n\| < \varepsilon$ for $n, m > N$.

Thus, from $|F_n(a,b) - F_m(a,b)| \leq \|F_n - F_m\|\|a,b\|$, it follows that $\{F_n(a,b)\}$is a real Cauchy sequence for all $(a,b) \in X \times X$.

Define $F(a,b) = \lim_{n\to\infty} F_n(a,b)$. Then we have

$$F(a+c,b+d) = \lim_{n\to\infty} F_n(a+c,b+d) = \lim_{n\to\infty}[F_n(a,b) + F_n(a,d) + F_n(c,b) + F_n(c,d)]$$
$$= \lim_{n\to\infty} F_n(a,b) + \lim_{n\to\infty} F_n(a,d) + \lim_{n\to\infty} F_n(c,b) + \lim_{n\to\infty} F_n(c,d)$$
$$= F(a,b) + F(a,d) + F(c,b) + F(c,d).$$
$$F(\alpha a, \beta b) = \lim_{n\to\infty} F_n(\alpha a, \beta b) = \lim_{n\to\infty} F_n\ \alpha\beta\ F_n(a,b) = \alpha\beta \lim_{n\to\infty} F_n(a,b) = \alpha\beta\ F(a,b).$$

Therefore, F is a linear 2-functional. Since we have$|\|F_n\| - \|F_m\|| \leq \|F_n - F_m\|$,

it follows that $\{\|F_n\|\}$ is a real Cauchy sequence.

Therefore, there is a real constant K such that $\|F_n\| \leq K$ for all n. $F \in X^*$since we have

$$|F(a,b)| = \left|\lim_{n\to\infty} F_n(a,b)\right| = \lim_{n\to\infty} |F_n(a,b)| \leq \lim_{n\to\infty} \|F_n\|\|a,b\| \leq K\ \|a,b\|.$$

Suppose $\|a,b\| \neq 0$.

For every $\varepsilon > 0$, there is an integer N such that $\|F_m - F_n\| < \varepsilon$ for $n, m > N$ and so we have $|F_m(a,b) - F_n(a,b)| \leq \|F_m - F_n\|\|a,b\| \leq \|a,b\|\ \varepsilon$ for all $n, m > N$.

Since $F(a,b) = \lim_{n\to\infty} F_n(a,b)$, there is an $M = M(a,b) > N$ such that

$|F_M(a,b) - F(a,b)| < \varepsilon\ \|a,b\|$.Therefore we have

$|F_n(a,b) - F(a,b)| \leq |F_n(a,b) - F_M(a,b)| + |F_M(a,b) - F(a,b)| \leq \varepsilon\ \|a,b\| + \varepsilon\ \|a,b\| = 2\ \|a,b\|\varepsilon$ for $n > N$. If $\|a,b\| = 0$, then $F_n(a,b) = 0 = F(a,b)$ andso we have

$|F_n(a,b) - F(a,b)| \leq 2\varepsilon\ \|a,b\|$.

Both cases yield that $|F_n(a,b) - F(a,b)| \leq 2\varepsilon\ \|a,b\|$ for all $n > N$, which means that $\|F_n - F\| \leq 2\varepsilon$ for $n > N$.Therefore $(X^*, \|\ .\ \|)$ is a Banach space.

4.Conclusion

I have presented some properties of the bounded liner 2-functionals in linear 2-normed spaces and related results to normed spaces and Banach spaces. One can also study Cauchy sequence, convergent sequence and completeness in liner 2- normed spaces.

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