Some Properties of Metric Spaces

Aye Aye Moe[1]

Abstract

In this paper, some definitions of countable set, uncountable set and metric space are firstly investigated. Then the properties of the countable set and uncountable set are studied. After that, the results of the metric spaces on a real line R, R^2 and the Euclidean space R^k are investigated. Moreover, some concepts of bounded set and compact set on the metric space are presented.

Introduction

In this paper, various notions of size associated to subsets of R will be investigated. Here on specific definitions and theorems of interest are concentrated on. A countable set is a set with the same cardinality (number of elements) as some subset of the set of natural numbers. An uncountable set (or uncountably infinite set) is an infinite set that contains too many elements to be countable. Here the elements of the theory of metric spaces; a metric space is simply a nonempty set X such that to each $x, y \in X$ there corresponds a non-negative number called the distance between x and y are given. In order to make the theory sufficiently rich, this distance is supposed to have certain properties, such as symmetry and the triangle inequality, that are familiar from Euclidean geometry.

1. Countable Set and Uncountable Set

In mathematics, a countable set is a set with the same cardinality (number of elements) as some subset of the set of natural numbers. Firstly we introduce the definitions of countable set and uncountable set. Then we will discuss some properties of countable set and uncountable set.

1.1 Definition. Let A and B be nonempty sets. If there exists a 1-1 mapping of A onto B, we can say that A and B can be put in 1-1 correspondence or that A and B have the same cardinal number or, briefly, $A \sim B$.

This relation clearly has the following properties:

(i) Reflexive : $A \sim A$,
(ii) Symmetric : If $A \sim B$, then $B \sim A$,
(iii) Transitive : If $A \sim B$ and $B \sim C$ then $A \sim C$.

Any relation with these three properties is called an ***equivalent relation***.

1.2 Definitions. For any positive integer n, let J_n be the set whose elements are the integers 1,2,3, …, n ; let J be the set consisting of all positive integers.

For any set A,

(a) A is ***finite*** if $A \sim J_n$ for some n.
(b) A is ***infinite*** if A is not finite.
(c) A is ***countable*** if $A \sim J$.
(d) A is ***uncountable*** if A is neither finite nor countable.

---

[1] Lecturer, Department of Mathematics, Dagon University

(e) *A* is ***at most countable*** if *A* is finite or countable.

We can describe a set is countable,
Formally, we have to show that there exists a bijectivefunction between the set in question and the set of natural numbers.
Informally, we have to show the elements of the set can be put in an order, so that
(i) no element will ever be repeated.
(ii) no element will even be missed out.

1.3 Examples. The set of **integers**

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is countable.

We can count the elements like this:

$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	…
$\mathbb{Z}$	0	1	–1	2	–2	3	–3	4	–4	5	…

The set of **rational numbers**,

$$Q = \{\frac{a}{b} : a, b, \in \mathbb{Z},\ b \neq 0\}$$

is countable.

$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	…
Q	0	$\frac{1}{1}$	$-\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	…

$\mathbb{N}$	11	12	13	14	15	16	17	18	19	20	…
Q	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{3}{2}$	$-\frac{3}{1}$	$\frac{4}{1}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	…

The set of **real numbers**, $\mathbb{R}$ is uncountable.

Suppose we have a "complete" list of the real numbers, written in decimal form

Cantor's Diagonal Method

0. $\boxed{3}$ 1 4 5 9 8 …
0. 9 $\boxed{2}$ 1 0 3 4 …
0. 1 0 $\boxed{9}$ 3 2 7 …
0. 5 9 1 $\boxed{5}$ 3 6 …
0. 7 2 9 3 $\boxed{8}$ 1 …
0. 2 7 3 4 5 $\boxed{3}$ …

We can create a new real number by adding 1 to each of the highlighted digits…

Cantor's Diagonal Method

0.	$\boxed{3}$	1	4	5	9	8	…	**4**
0.	9	$\boxed{2}$	1	0	3	4	…	**3**
0.	1	0	$\boxed{9}$	3	2	7	…	**0**
0.	5	9	1	$\boxed{5}$	3	6	…	**6**
0.	7	2	9	3	$\boxed{8}$	1	…	**9**
0.	2	7	3	4	5	$\boxed{3}$	…	**4**

The number **0.430694** will never appear in our list (no matter how long it goes on.)

Its 1st digit is different from the 1st digit of the 1st number in the list.
Its 2nd digit is different from the 2nd digit of the 2nd number in the list.
…
Its 100th digit is different from the 100th digit of the 100th number in the list; etc…

The elements of $\mathbb{R}$ cannot be placed in one-to-one correspondence with the element of $\mathbb{N}$.
So $\mathbb{R}$ is uncountable.

1.4 Theorem. Let A and B be two countable sets. Then the Cartesian product $A \times B$ is countable.

Proof: There are three cases to consider.

Case 1: If both A and B are finite with $|A|$ and $|B|$ then it is easy to show that $|A| \times |B| = mn$ and hence $A \times B$ is finite and so it is countable.

Case 2: If A is finite with $|A| = n$ and B is countably infinite, then there exists bijective functions

$$f : A \to \{1, 2, \ldots, n\} \text{ and } g : B \to N.$$

Define $h : A \times B \to N$ for all $(a,b) \in A \times B$ by

$$h(a,b) = 2^{f(a)} \cdot 3^{g(b)}.$$

Then clearly h is injective by the unique factorization of the each natural number.
So $A \times B$ is countable.

Case 3: If A and B are both countable then there exists bijective functions $f : A \to N$ and $g : B \to N$ and defining $h : A \times B \to N$ as above still gives us an injective function, and so $A \times B$ is countable. □

1.5 Proposition. The algebraic numbers A are countable.

Proof: Since all algebraic numbers (including complex numbers) are roots of a polynomial.

Let the polynomial be $a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n$, and the algebraic number α is the k^{th} root of the polynomial.

We can define an injection function

$f : A \to Q$ given by

$$f(\alpha) = 2^{k-1} \cdot 3^{a_0} \cdot 5^{a_1} \cdot 7^{a_2} \cdot \ldots \cdot p_{n+2}^{\;a_n}$$, while p_n is the n^{th} prime. □

1.6 Theorem. Let A be a non-empty set. Then the following are equivalent

(a) A is countable.

(b) There exists a surjection $f:\mathbb{N}\to A$.

(c) There exists an injection $g:A\to\mathbb{N}$.

Proof. (a)$\Rightarrow$(b). If A is countably infinite, then there exists a bijection $f:\mathbb{N}\to A$ and then (b) follows: If A is finite, then there is a bijection $h:\{1,2,\ldots,n\}\to A$ for some n. Then the function $f:\mathbb{N}\to A$ defined by

$$f(i)=\begin{cases} h(i) & 1\le i\le n \\ h(n) & i>n \end{cases}$$

is a surjection .

(b)$\Rightarrow$(c). Assume that $f:\mathbb{N}\to A$ is a surjection.We claim that there is an injection $g:A\to\mathbb{N}$. To define g note that if $a\in A$, then $f^{-1}(\{a\})\neq\phi$. Hence we set $g(a)=\min\ f^{-1}(\{a\})$. Now note that if $a\neq a'$, then the sets $f^{-1}(\{a\})\cap f^{-1}(\{a'\})=\phi$ which implies $\min^{-1}(\{a\})\neq\min^{-1}(\{a'\})$. Hence $g(a)\neq g(a')$ and $g:A\to\mathbb{N}$ is an injective.

(c)$\Rightarrow$(a). Assume that $g:A\to\mathbb{N}$ is an injective. We want to show that A is countable. Since $g:A\to g(A)$ is a bijection and $g(A)\subset\mathbb{N}$. Any subset of countable set is countable implies that A is countable.□

1.7 Theorem. Let $\{E_n\}, n=1,2,3,\ldots$ be a sequence of countable sets and put $S=\bigcup_{n=1}^{\infty}E_n$. Then S is countable.

Proof. Since E_n, n = 1,2,3,… are countable, we can arrange the elements of E_n in a sequence $\{x_{nk}\}, k=1,2,3,\ldots$ for each n.

Consider the following infinite array in which the elements of E_n from the n^{th} row

$$\begin{array}{llllll}
E_1: & x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\
E_2: & x_{21}\nearrow & x_{22}\nearrow & x_{23}\nearrow & x_{24} & \cdots \\
E_3: & x_{31}\nearrow & x_{32}\nearrow & x_{33} & x_{34} & \cdots \\
E_4: & x_{41}\nearrow & x_{42} & x_{43} & x_{44} & \cdots \\
\vdots & & & & & \\
E_n: & x_{n1} & x_{n2} & x_{n3} & x_{n4} & \cdots
\end{array}$$

The array contains all elements of S and these elements can be arranged in a sequence.

$x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}\ldots$.

If any two of the set E_n have common elements, these will appear more than once in the above sequence.

Hence there is a subset T of the set of all positive integers such that $S\sim T$.

Since the set of all positive integers is countable, T is at most countable.

Thus S is at most countable.

Since $E_1\subset S$, and E_1 is infinite, S is infinite, and thus countable. □

1.8 Theorem. Let *A* be the set of all sequences whose elements are the digit 0 and 1. This set *A* is uncountable.

Proof. The elements of *A* are sequences like $1, 0, 0, 1, 0, 1, 1, 0, 1, \ldots$.
Let *E* be a countable subset of *A* and let E consist of the sequences $s_1, s_2, s_3, \ldots$.
Construct a sequence *s* as follows:
If the n^{th} digit in s_n is 1, let the n^{th} digit of *s* be 0; and vice versa.
Then the sequence *s* differs from every member of *E* in at least one place.
Hence $s \notin E$. But $s \in A$.
Thus *E* is a proper subset of *A*.
This shows that every countable subset of *A* is a proper subset of *A*.
If *A* is countable, *A* will be a proper subset of *A*, and it is impossible.
Hence *A* is uncountable. □

2. Metric Spaces

In mathematics, a metric space is a set for which distance between all members of the set are defined. Those distances, taken together, are called a metric on the set. A metric on a space includes topological properties like open and closed sets, which lead to the study of more abstract topological spaces.

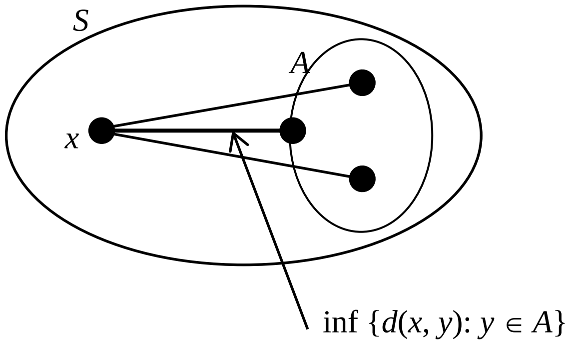

Fig. 2.1

2.1 Definitions. A set *X*, whose elements we shall call ***points***, is said to be a ***metric space*** if with any two points *p* and *q* of *X* there is associated a real number $d(p,q)$, called the ***distance*** from *p* to *q*; such that

(i) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$ if $p = q$

(ii) $d(p,q) = d(q,p)$

(iii) $d(p,q) \leq d(p,r) + d(r,q)$, for any $r \in X$.

Any function with these three properties is called a ***distance function***, or ***a metric***.

2.2 Theorem. Every neighbourhood is an open set.

Proof. Let $E = N_r(p)$ be a neighbourhood of a point *p* with radius *r*.
Let *q* be any point of *E*.

Then $d(p,q) < r$.
Let $h = r - d(p,q) > 0$.
Then $d(p,q) = r - h$.

For all points *s* such that $d(q,s) < h$, then

$$d(p,s) \leq d(p,q) + d(q,s)$$
$$< r - h + h$$
$$= r.$$

So $s \in N_r(p) = E$.
Thus *q* is an interior point of *E*.

Hence *E* is open. ▫

2.3 Theorem. If *f* is a continuous mapping of a compact metric space *X* into R^k, then $f(X)$ is closed and bounded. Thus, *f* is bounded.

Proof. Let f be a continuous mapping of a compact metric space *X* into R^k.
$f(X)$ is compact in R^k.
$f(X)$ is closed and bounded.

Next to prove *f* is bounded.

Let $x \in X$ and $f(x) = (f_1(x), \ldots, f_k(x))$.

Since $f(X)$ is bounded, $\exists N \in R, \exists\, y = (y_1, \ldots, y_k) \in R^k$ such that $|f(x) - y| < N$.

Now, $$|f_i(x) - y_i| \leq \left\{ \sum_{i=1}^{k} [f_i(x) - y_i]^2 \right\}^{1/2}$$
$$= |f(x) - y| < N.$$

Hence $-N < f_i(x) - y_i < N$, for $i = 1, 2, \ldots, k$.

Thus $-N + y_i < f_i(x) < N + y_i = M$ (say),so that

$$-M < f_i(x) < M \quad \text{for } i = 1, 2, \ldots, k.$$

Thus $$|f(x)| \leq \left\{ \sum_{i=1}^{k} (f_i(x))^2 \right\}^{1/2} < (kM^2)^{1/2}.$$

That is, $|f(x)| \leq L$ for all $x \in X$.

Hence *f* is bounded. ▫

2.4 Definitions. An ***open cover*** of a set *E* in a metric space *X* is a collection $\{G_\alpha\}$ of open subsets of *X* such that $E \subset \bigcup_{\alpha} G_\alpha$.

A subset *K* of a metric space *X* is said to be ***compact*** if every open cover of *K* contains a finite subcover.

2.5 Theorem. Compact subsets of metric spaces are closed.

Proof. Let *K* be a compact subset of a metric space.
Suppose $p \in X, p \notin K$.

If $q \in K$, let V_q and W_q be neighbourhoods of *p* and *q*, respectively, of radius less than $\frac{1}{2}d(p,q)$.

Since *K* is compact, there are finitely many points $q_1,\dots,q_n$ in *K* such that

$K \subset W_{q_1} \cup \dots \cup W_{q_n} = W$.

If $V = V_{q_1} \cap \dots \cap V_{q_n}$, then *V* is a neighbourhood of *p* which does not intersect *W*.

Hence $V \subset K^c$, so that *p* is an interior point of K^c.
Thus K^c is open.

Hence *K* is closed. □

Conclusion

Mathematical analysisisthe branch of mathematics dealing with limit and related theories such as differentiation, integration, measure, infinite series, and analytic function. In this paper, these theories are usually studied in the context of real and complex numbers and function. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has definition of nearness (a topological space) or special distance between objects (a metric space).

Acknowledgements

I would like to express my sincere gratitude to Dr. U Win Naing, Rector of Dagon University for his permission to undertake this paper. I am greatly indebted to Professor Dr. U Thein Aung, Head of the Department of Mathematics, Dagon University for his encouragement, and valuable comments. I am also indebted to Dr. Daw Hla Hla Kyi, Professor, Department of Mathematics, Dagon University for her support and suggestions.

References

[1] Online.

[2] W. Rudin, *Principles of Mathematical Analysis*, Second Edition, McGraw-Hill Inc., New York, 1976.

[3] G. F. Simmons, *Introduction to Topology and Modern Analysis*, International Student Edition, McGraw-Hill Inc., New York, 1963.