A study on Fundamental Relations of Traffic Flow

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Abstract

In this paper, the fundamental relations for a traffic flow are studied. First, the fundamental relations between flow, density and speed are introduced. Next, the relation between time mean speed and space mean speed are discussed.

Keywords: Fundamental relations, traffic flow, density, speed

Introduction

The traffic congestion is one of the major problems in mega cities. In this section, we will introduce the basic definitions, fundamental relations, and the concepts of traffic flow.

Definition: ***Flow q*** is defined as the number of vehicles per unit time passing a point in space, and is usually expressed in vehicles in per hour.

Definition: ***Speed v*** is the rate of change of distance with respect to time, and is usually measured in either mile per hour or kilometers per hour or feet per second or meters per second, depending on the application.

Definition: ***Density k*** is defined as the number of vehicles per unit distance occupying a section of roadway at a given instant in time, and is usually measured in vehicles per mile or vehicles per kilometer.

Basic Parameters of Traffic Flow

In the traffic theory, speed is one of the basic parameters of traffic flow. Here, the two representations of speed are *time mean speed* and *space mean speed*. In this section, time mean speed and space mean speed are defined and the relationship between them is discussed then the fundamental parameters of traffic flow will also be derived. In addition, this relationship can be represented in graphical form resulting in the fundamental diagrams of traffic flow.

2 **Time Mean Speed and Space Mean Speed**

Time mean speed is the average of all vehicles passing a point over duration of time. It is the average (arithmetic mean) of spot speed. The ***time mean speed*** v_t is given by,

$$v_t = \frac{1}{n}\sum_{i=1}^{n} v_i \tag{1}$$

where v_i is the *spot speed of the* i^{th} *vehicle,* and n is the number of observations.

Speeds are represented in the form of frequency table. Then, the ***time mean speed*** is given by,

$$v_t = \frac{\sum_{i=1}^{N} f_i\, v_i}{\sum_{i=1}^{N} f_i} = \frac{1}{n}\sum_{i=1}^{N} f_i\, v_i \tag{2}$$

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where f_i is the number of vehicles having speed v_i, and N is the number of speed classes. Here $n=\sum_{i=1}^{N} f_i$. The *space mean speed* is also an average of the spot speed, but spatial weightage is given instead of temporal. This is derived as follows. Consider unit length of a road, and let v_i is the spot speed of i^{th} vehicle. Let t_i be the time the vehicle takes to complete unit distance, then the average time is given by

(3) $$t_s = \frac{1}{n}\sum_{i=1}^{n} t_i = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{v_i}.$$

From the above equation, we have the ***space mean speed***

(4) $$v_s = \frac{1}{t_s} = n\left(\sum_{i=1}^{n}\frac{1}{v_i}\right)^{-1}.$$

We can easily see that the space mean speed is the *harmonic mean of spot speed*.

Example. If the spot speeds are 65, 62, 58, 55, 50, 48 and 45 (ft/s), then the time mean speed is v_t = (65+62+58+55+50+48+45) / 7 = 54.71 ft/s, and the space mean speed is

$$v_s = 7\times\left(\frac{1}{65}+\frac{1}{62}+\frac{1}{58}+\frac{1}{55}+\frac{1}{50}+\frac{1}{48}+\frac{1}{45}\right)^{-1} = 53.85 \text{ ft/s}.$$

If the spot speeds are expressed as a frequency table, then, form equation 1.5 (2)

(1) $$v_s = \frac{\sum_{i=1}^{N} f_i}{\sum_{i=1}^{N}\frac{f_i}{v_i}} = n\left(\sum_{i=1}^{N}\frac{f_i}{v_i}\right)^{-1}$$

where f_i vehicle will have v_i speed, n is the number of speed and N is the number of class.

In the following table, average speed is computed, which is the mean of the speed range. The volume of flow f_i for that speed range is the same as the frequency.

Class (i)	speed-range (m/s)	average-speed v_i	volume of flow f_i	$f_i\ v_i$	f_i / v_i
1	3 - 6	4.5	1	4.5	0.2222
2	7 - 10	8.5	4	34.0	0.4706
3	11 - 14	12.5	7	87.5	0.5600
4	15 - 18	16.5	9	148.5	0.5455
		Total =	n = 21	274.5	1.7983

Here, number of classes is N = 4.

Time mean speed $$v_t = \frac{\sum_{i=1}^{N} f_i v_i}{n} = \frac{274.5}{21} = 13.0714 \text{m/s}.$$

Space mean speed $$v_s = \frac{n}{\sum_{i=1}^{N}\frac{f_i}{v_i}} = \frac{21}{1.7983} = 11.6777 \text{ m/s}.$$

In order to understand the concept of time mean speed and space mean speed, let us consider the following example.

Example. Let there be a road stretch having two sets of vehicle as in Fig. 1. The first vehicle is traveling at 10 m/s with 50 m spacing, and the second set at 20 m/s with 100 m spacing. Therefore, the *headway* of the slow vehicle $h_1 = 50$ m $\div 10$ m/s = 5 sec. The number of slow-moving vehicles observed at **A** in *one minute* will be $f_1 = 60/5 = 12$ vehicles.

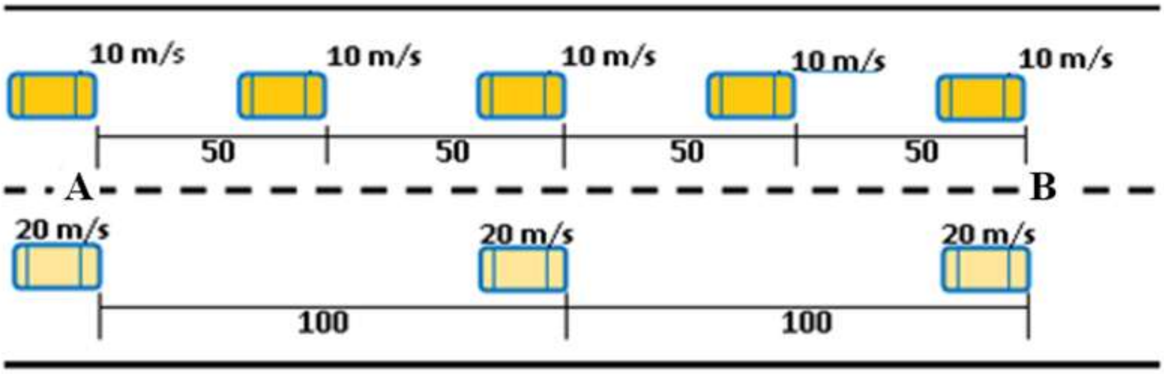

Figure 1 Illustration of relation between time mean speed and space mean speed

The density k is the number of vehicles in 1 km, and is the inverse of spacing.

4 Therefore, for slow vehicles $k_1 = 1000 \div 50 = 20$ veh / km.

For the fast vehicle, $h_2 = 100 \div \mathbf{20}$ m/s = 5 s,

$f_2 = 60/5 = 12$ vehicles and the density $k_2 = 1000 \div 100 = \mathbf{10}$ veh/ km.

Therefore, by definition, time mean speed v_t is given by

$$v_t = (f_1 . v_1 + f_2 . v_2) / (f_1 + f_2) = (12\times10+12\times20) / (12+12) = 15 \text{ m/s}.$$

Similarly, by definition, space mean speed is the mean of vehicle speeds over time. Therefore,

$$v_s = (k_1 v_1 + k_2 v_2) / (v_1 + v_2) = (20\times10+10\times20) / (10+20) = 13.3 \text{ m/s}.$$

This is same as the harmonic mean of spot speeds obtained at location A,

$$v_s = (f_1 + f_2) / (f_1 / v_1 + f_2 / v_2) = 24 / (12/10 + 12/20) = 13.3 \text{ m/s}.$$

It may be noted that since harmonic mean is always lower than the arithmetic mean and we observe that *space mean speed is always lower than the time mean speed*.

The Fundamental Relations and Diagrams

In this section, we will present the fundamental relations between traffic variables and useful fundamental diagrams.

Fundamental Relation of a Traffic Flow

The relationship between the fundamental variables of traffic flow, speed, and density is called the fundamental relation of the traffic flow. This can be derived as follow:

Let the length road be v km, and assume all the vehicles are moving with v km / hr.

Let the number of vehicles counted by an observer at **A** for one hour be n_1.

By definition, the number of vehicles counted in one hour is *flow* (q). Therefore,

(1) $$n_1 = q.$$

Similarly, by definition, *density* (k) is the number of vehicles in unit distance. Therefore, number of vehicles n_2 in a road stretch of distance v *km* will be density multiplied by distance. So

(2) $$n_2 = k \times v.$$

Since all the vehicles have speed v, the number of vehicles counted in 1 hour and the number of vehicles in the stretch of distance v *km* will also be the same. Therefore,

(3) $$q = k \times v.$$

This is the fundamental equation of a traffic flow.

Note that, v in the above equation refers to the space mean speed v_s.

Relation between Time Mean Speed and Space Mean Speed

Consider a stream of vehicles with a set of sub-stream flow $q_1, q_2, \ldots, q_n$ having speed

$v_1, v_2, \ldots, v_n$. The fundamental relation between flow (q), density (k), and mean speed (v_s) is

(1) $$q = k \times v_s$$

where $q = \sum_{i=1}^{n} q_i$, $k = \sum_{i=1}^{n} k_i$ and v_s is the space mean speed.

Therefore for any sub-stream q_i , the following relationship will be valid.

(2) $$q_i = k_i \times v_i \,.$$

Let f_i denote *the proportion of sub-stream density* k_i to the total density k,

(3) $$f_i = \frac{k_i}{k}.$$

Space mean speed averages the speed over *space*. Therefore, if k_i vehicle has v_i speed, then space mean speed is given by

(4) $$v_s = \frac{\sum_{i=1}^{n} k_i v_i}{k} \quad .$$

Time mean speed averages the speed over *time*. Therefore,

(5) $$v_t = \frac{\sum_{i=1}^{n} k_i v_i^2}{q} \quad .$$

Substituting (2) in (5), v_t can be written as,

(6) $$v_t = \frac{\sum_{i=1}^{n} q_i v_i}{q} \quad .$$

Substituting (1) and (3) in (5), we get the relation

(7) $$v_t = \frac{k\sum_{i=1}^{n}\frac{k_i}{k}v_i^2}{q} = \frac{\sum_{i=1}^{n} f_i v_i^2}{v_s}$$

and it can also be written as

6 $$v_t = \frac{\sum_{i=1}^{n} f_i (v_s + (v_i - v_s))^2}{v_s} = \frac{1}{v_s}\left[\sum_{i=1}^{n} f_i \ (v_s)^2 + \sum_{i=1}^{n} f_i \ (v_i - v_s)^2 + 2v_s \sum_{i=1}^{n} f_i \ (v_i - v_s)\right].$$

The third term of the equation will be zero because $f_i(v_i - v_s)$ will be zero, since v_s is the mean speed of v_i. The numerator of the second term gives the standard deviation of v_i. $v_t = v_s \sum f_i + \frac{\sigma^2}{v_s}$, by definition, $\sum f_i = 1$. Therefore $v_t = v_s + \frac{\sigma^2}{v_s}$.

Hence, time mean speed will be always greater than space mean speed since variance cannot be negative and hence $v_t > v_s$.

If all the speed of the vehicles is the same, then spot speed, time mean speed and space mean speed will also be the same.

Fundamental Diagrams

The relation between flow and density, density and speed, speed and flow, can be represented with the help of some curves. They are referred to the fundamental diagrams of traffic flow. The relation between the density and the corresponding flow on a given stretch of road is referred to as one of the fundamental diagram of traffic flow. Some characteristics of an ideal flow-density relationship is listed below:

(1) If $k = 0$ then $q(0) = 0$.

(2) If $k = k_{max}$ is the maximum density, then $q(k_{max}) = 0$.

(3) If $k = k_c$ is a critical point of the density, then $q(k_c) = q_{max}$ where q_{max} is the maximum flow.

(4) The relationship is normally represented by a parabolic curve as shown in Fig.2

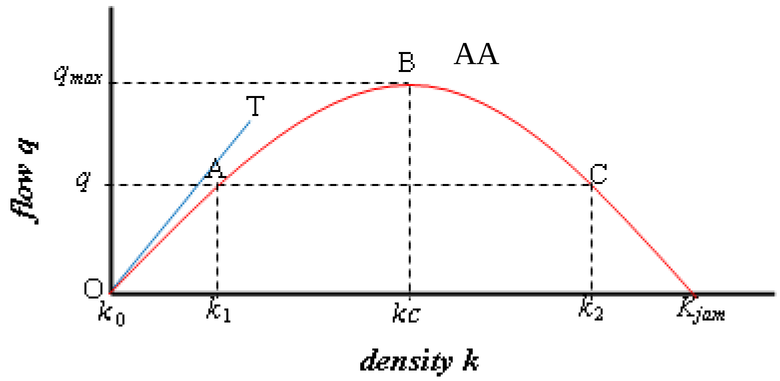

Figure 2 Flow-density diagram

In Fig. 2, the point B refers to the maximum flow q_{max} and the corresponding density is k_c. The point D refers to the maximum density k_{jam} and the corresponding flow is zero. OT is the tangent drawn to the parabola at O, and the slope of the line OT gives the *mean free flow* *speed* (the speed with which a vehicle can travel when there is no flow). Points A and C correspond to same flow but have two different densities. Further, the slope of the line OA gives the mean speed at density k_1 and slope of the line OC will give mean speed at density k_2. Clearly the speed at density k_1 will be higher since there is less number of vehicles on the road.

Similar to the flow-density relationship, speed will be maximum, referred to as the free flow speed, and when the density is maximum the speed will be zero. If density is nearly zero k_0, vehicles will be flowing with their desired speed v_f, or free flow speed. When the density is jam density k_{jam}, the speed of the vehicles becomes zero. It is also possible to have non-linear relationships. The flow is zero either because there is no vehicle or there are too many vehicles so that they cannot move. At maximum flow, the speed will be in between zero and free flow speed.

Conclusion

The fundamental relations and the basic parameters discussed in this paper are essential to solving traffic congestion. Moreover, we can conclude from our observations in this paper that in many fundamental traffic equations, space mean speed is more preferable than time mean speed because space mean speed weights slower vehicles more heavily as they occupy the road stretch for longer duration of time.

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References

[1] S. Chanut and C. Buisson, Macroscopic Model and Its Numerical Solution for Two-Flow Mixed Traffic with Different Speeds and Lengths, Transportation Research Record 1852, Paper No. 03-2472.

[2] M. J. Lighthill and G. B. Whitham, On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, May 1955.

[3] S. Logghe & L.H. Immers, *Multi-class kinematic wave theory of traffic flow*, Available online at www.sciencedirect.com, Transportation Research Part B 42 (2008) 523–541.

[4] T.V. Mathew and K.V. Krishna Rao, *Introduction to Transportation Engineering*, NPTEL May 3, 2007. 30.1 - 30.7, 31.1 - 31.7. t Co.lit, New York, 1961.